67 ※ Derivatives ※ (Chapter no. 2) ①

Derivative of a function:- Let $f(x)$ be a real valued function then derivative of $f(x)$ denoted by $f'(x)$ & is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Some standard formulae of differentiation:- Set ①

① $\frac{d}{dx}(x^n) = nx^{n-1}$

② $\frac{d}{dx}(c) = 0$ where c is any Constt.

③ $\frac{d}{dx}(x) = 1$

④ $\frac{d}{dx}(a^{bx}) = a^{bx}.b.\ln a$

⑤ $\frac{d}{dx}(e^{ax}) = ae^{ax}$

⑥ $\frac{d}{dx}(\ln x) = \frac{1}{x}$

⑦ $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}.a$

⑧ $\frac{d}{dx}(u\pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

⑨ $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$

⑩ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$

⑪ $\frac{d}{dx}(cu) = c.\frac{du}{dx}$

Set (2)

(2)

(1) $\frac{d}{dx}(\sin x) = \cos x$

(2) $\frac{d}{dx}(\cos x) = -\sin x$

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$

(4) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(5) $\frac{d}{dx}(\sec x) = \sec x.\tan x$

(6) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$

Set (3)

(1) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(3) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(4) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(5) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(6) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Set (4)

(1) $\frac{d}{dx}(\sinh x) = \cosh x$

(2) $\frac{d}{dx}(\cosh x) = \sinh x$

(3) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(4) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(5) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x.\tanh x$

(6) $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x.\coth x.$

③

Set ⑤

① $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

② $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

③ $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$ $\quad |x| < 1$

④ $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$ $\quad |x| > 1$

⑤ $\frac{d}{dx}(\operatorname{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}$

⑥ $\frac{d}{dx}(\operatorname{csch}^{-1}x) = \frac{-1}{x\sqrt{1+x^2}}$

4

EXERCISE 2.2

Differentiate w.r.t. x, (Problems 1 – 14)

1. $\sqrt{a^2+x^2}$

Sol. Let $y = \sqrt{a^2+x^2}$

or $y = (a^2+x^2)^{1/2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}(a^2+x^2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(a^2+x^2).$$

5

$$\frac{dy}{dx} = \frac{1}{2}(a^2+x^2)^{-1/2}\,2x$$

$$= \frac{x}{(a^2+x^2)^{1/2}}$$

$$= \frac{x}{\sqrt{a^2+x^2}}$$

2. $\sqrt[3]{x^2+x+1}$

Sol. Let $y = (x^2+x+1)^{1/3}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{3}(x^2+x+1)^{1/3-1}\cdot\frac{d}{dx}(x^2+x+1)$$

$$= \frac{1}{3}(x^2+x+1)^{-2/3}\cdot(2x+1)$$

$$= \frac{1}{3}\cdot\frac{1}{(x^2+x+1)^{2/3}}\cdot(2x+1)$$

$$= \frac{2x+1}{3(x^2+x+1)^{2/3}}$$

3. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$

Sol. Let $y = \dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$

Multiplying numerator & denominator by $\sqrt{a+x}-\sqrt{a-x}$

$$= \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}} \times \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$$

$$= \frac{(\sqrt{a+x}-\sqrt{a-x})^2}{(\sqrt{a+x})^2-(\sqrt{a-x})^2} = \frac{(a+x)+(a-x)-2\sqrt{a+x}\sqrt{a-x}}{(a+x)-(a-x)}$$

$$= \frac{2a-2\sqrt{(a+x)(a-x)}}{a+x-a+x} = \frac{2a-2\sqrt{a^2-x^2}}{2x}$$

$$y = \frac{a-\sqrt{a^2-x^2}}{x}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{x\left[0-\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\right]-\left[a-\sqrt{a^2-x^2}\right]\cdot 1}{x^2}$$

$$= \frac{x\cdot\frac{x}{\sqrt{a^2-x^2}}-a+\sqrt{a^2-x^2}}{x^2}$$

$$= \frac{\frac{x^2}{\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}-a}{x^2} = \frac{\frac{x^2+a^2-x^2}{\sqrt{a^2-x^2}}-a}{x^2}$$

⑥

$$= \frac{\frac{a^2}{\sqrt{a^2-x^2}} - a}{x^2} = \frac{a^2 - a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}} = \frac{a(a-\sqrt{a^2-x^2})}{x^2\sqrt{a^2-x^2}}$$

4. $y = \frac{\sqrt{\sin x}}{\sin\sqrt{x}}$

Sol.- Let $y = \frac{\sqrt{\sin x}}{\sin\sqrt{x}}$

Diff. w.r.t. x

Sol.-

$$\frac{dy}{dx} = \frac{\sin\sqrt{x}\left\{\frac{1}{2\sqrt{\sin x}} \cdot \cos x\right\} - \sqrt{\sin x}\left\{\cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\right\}}{(\sin\sqrt{x})^2}$$

$$= \frac{\sin\sqrt{x} \cdot \frac{\cos x}{2\sqrt{\sin x}} - \frac{\sqrt{\sin x} \cdot \cos\sqrt{x}}{2\sqrt{x}}}{(\sin\sqrt{x})^2}$$

$$= \frac{\frac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}}}{(\sin\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}\sin^2\sqrt{x}}$$

5. $\sqrt{\log_{10}(x^2+1)}$

Sol. Let $y = \sqrt{\log_{10}(x^2+1)}$

or $y = \sqrt{\frac{\log_e(x^2+1)}{\log_e 10}} = \sqrt{\frac{\ln(x^2+1)}{\ln 10}} = \frac{\sqrt{\ln(x^2+1)}}{\sqrt{\ln 10}}$

or $y = \frac{1}{\sqrt{\ln 10}} \cdot \sqrt{\ln(x^2+1)}$

Diff. w.r.t. x

$$dy/dx = \frac{1}{\sqrt{\ln 10}} \cdot \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot 2x$$

$$\Rightarrow dy/dx = \frac{x}{\sqrt{\ln 10} \cdot \sqrt{\ln(x^2+1)} \cdot (x^2+1)} \text{ Ans.}$$

6. $y = \tan(\sin x)$

Diff. w.r.t. x

Sol. $\frac{dy}{dx} = \sec^2(\sin x) \cdot \frac{d}{dx}(\sin x)$

$= \sec^2(\sin x) \cdot \cos x$

$= \cos x \cdot \sec^2(\sin x)$.

Note $\frac{\log_a x}{\log_a y} = \log_y x$.

7. $y = \arc\tan\left(\dfrac{x\sin a}{1-x\cos a}\right)$

Diff. w.r.t. x

Sol. $\dfrac{dy}{dx} = \dfrac{1}{1+\left(\dfrac{x\sin a}{1-x\cos a}\right)^2}\cdot\dfrac{d}{dx}\left(\dfrac{x\sin a}{1-x\cos a}\right)$

$= \dfrac{1}{1+\dfrac{(x\sin a)^2}{(1-x\cos a)^2}}\cdot\dfrac{(1-x\cos a).\sin a - x\sin a.(-\cos a)}{(1-x\cos a)^2}$

$= \dfrac{1}{\dfrac{(1-x\cos a)^2+(x\sin a)^2}{(1-x\cos a)^2}}\cdot\dfrac{\sin a - \cancel{x\sin a\cos a} + \cancel{x\sin a\cos a}}{(1-x\cos a)^2}$

$=$

$= \dfrac{\sin a}{1-2x\cos a + x^2\cos^2 a + x^2\sin^2 a} = \dfrac{\sin a}{1-2x\cos a + x^2(\cos^2 a+\sin^2 a)}$

$= \dfrac{\sin a}{1-2x\cos a + x^2}$

8. $y = \ln\dfrac{x^2+x+1}{x^2-x+1}$

Sol. $y = \ln(x^2+x+1) - \ln(x^2-x+1)$

Diff. w.r.t. x

$\dfrac{dy}{dx} = \dfrac{1}{(x^2+x+1)}\dfrac{d}{dx}(x^2+x+1) - \dfrac{1}{(x^2-x+1)}\dfrac{d}{dx}(x^2-x+1)$

$= \dfrac{2x+1}{x^2+x+1} - \dfrac{2x-1}{x^2-x+1}$

$= \dfrac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)}$

$= \dfrac{(2x^3-2x^2+2x+x^2-x+1)-(2x^3+2x^2+2x-x^2-x-1)}{(x^2+x+1)(x^2-x+1)}$

$= \dfrac{2x^3-x^2+x+1-2x^3-x^2-x+1}{(x^2+x+1)(x^2-x+1)} = \dfrac{-2x^2+2}{(x^2+x+1)(x^2-x+1)}$

$= \dfrac{2(1-x^2)}{(x^2+x+1)(x^2-x+1)}$

9. $y = x^{x^2}$

Sol. Let $y = x^{x^2}$

taking log on both sides

$\ln y = \ln x^{x^2}$

$\ln y = x^2 \ln x$

Diff. w.r.t. x

$\dfrac{1}{y}.\dfrac{dy}{dx} = x^2.\dfrac{1}{x} + \ln x . 2x$

$$= x + 2x\ln x$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\,(1 + 2\ln x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\,.\,x\,(2\ln x + 1)$$

$$= x^{x^2}\cdot \; x\,(2\ln x + 1)$$

$$= x^{x^2+1}\cdot \; (2\ln x + 1)$$

10. $\ln(x^2 + x)$

Sol. Let $y = \ln(x^2 + x)$

Differentiating w.r.t x, we have

$$\frac{dy}{dx} = \frac{1}{(x^2 + x)}\cdot\frac{d}{dx}(x^2 + x)$$

$$= \frac{1}{x^2 + x}\cdot(2x+1)$$

$$= \frac{2x+1}{x(x+1)}$$

11. $y = (\arcsin x)^{x^{\frac{1}{x}}}$

Sol. Taking logarithm of both sides

$$\ln y = \ln(\sin^{-1}x)^{x^{1/x}}$$

$$\ln y = x^{1/x}\cdot\ln(\sin^{-1}x)$$

Diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x^{1/x}\cdot\frac{1}{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}} + \ln\sin^{-1}x\cdot\frac{d}{dx}\left(x^{\frac{1}{x}}\right) \quad \text{---} (1)$$

Now; let $u = x^{1/x}$

$$\ln u = \ln x^{1/x}$$

or $\ln u = \frac{1}{x}\cdot\ln x$

Diff. w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{x}\cdot\frac{1}{x} + \ln x\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2} + \ln x\left(-\frac{1}{x^2}\right)$$

$$\frac{du}{dx} = u\left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right)$$

$$= x^{1/x}\frac{1}{x^2}(1 - \ln x)$$

$$\frac{du}{dx} = x^{\frac{1}{x}-2}(1 - \ln x)$$

or $\frac{d}{dx}(x^{1/x}) = x^{\frac{1}{x}-2}(1-\ln x)$

Putting in (1)

$$\frac{1}{y}\frac{dy}{dx} = x^{\frac{1}{x}} \cdot \frac{1}{\sin^{-1}x\cdot\sqrt{1-x^2}} + \ln\sin^{-1}x \cdot x^{\frac{1}{x}-2}(1-\ln x)$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left[x^{\frac{1}{x}}\cdot\frac{1}{\sin^{-1}x\cdot\sqrt{1-x^2}} + x^{\frac{1}{x}-2}\cdot\ln\sin^{-1}x\cdot\ln(1-x)\right]$$

$$= (\sin^{-1}x)^{x^{\frac{1}{x}}}\left[x^{\frac{1}{x}}\cdot\frac{1}{\sin^{-1}x\cdot\sqrt{1-x^2}} + x^{\frac{1}{x}-2}\cdot\ln\sin^{-1}x\cdot\ln(1-x)\right]$$

12. $y = |x^2-9|$

Sol. Given $y = |x^2-9|$

Here $y = x^2-9, \quad$ if $|x| \geq 3$

$\quad = -(x^2-9), \quad$ if $|x| < 3$

Diff. w.r.t. x

$$\left.\begin{aligned}\frac{dy}{dx} &= 2x \quad \text{if } |x| \geq 3\\ &= -2x \quad \text{if } |x| < 3\end{aligned}\right] \text{— Ans.}$$

<u>Note</u> Let $x \in R$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0\end{cases}$$

13. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

Sol. Let $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\frac{d}{dx}\left(x+\sqrt{x+\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)\right)$$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\right)\cdot\left(1+\frac{1}{2\sqrt{x}}\right) \text{— Ans.}$$

14. $y = (x + |x|)^{1/2}$

Sol. Here, $y = (x+x)^{1/2}$, if $x \geq 0$
$= (x-x)^{1/2}$ if $x < 0$

or $y = (2x)^{1/2}$ if $x \geq 0$
$= 0$ if $x < 0$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}(2x)^{-1/2}\cdot 2 = \frac{1}{\sqrt{2x}} \quad \text{if } x \geq 0$$
$$= 0 \quad \text{if } x < 0$$

15. Differentiate

$$\arctan \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \quad \text{w.r.t.} \quad \arccos x^2$$

Sol. Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$ & $u = \cos^{-1}x^2$ then $\frac{dy}{du} = ?$

as $u = \cos^{-1}x^2 \Rightarrow x^2 = \cos u$. Put in eq.

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos u} - \sqrt{1-\cos u}}{\sqrt{1+\cos u} + \sqrt{1-\cos u}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\frac{u}{2}} - \sqrt{2\sin^2\frac{u}{2}}}{\sqrt{2\cos^2\frac{u}{2}} + \sqrt{2\sin^2\frac{u}{2}}}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cdot\cos u/2 - \sqrt{2}\sin u/2}{\sqrt{2}\cos u/2 + \sqrt{2}\sin u/2}\right]$$

$$= \tan^{-1}\left[\frac{\cos u/2 - \sin u/2}{\cos u/2 + \sin u/2}\right]$$

Dividing numerator & denominator by $\cos u/2$.

$$= \tan^{-1}\left(\frac{1-\tan u/2}{1+\tan u/2}\right)$$

$$y = \tan^{-1}\left[\frac{\tan \pi/4 - \tan u/2}{1+\tan \pi/4 \cdot \tan u/2}\right] = \tan^{-1}.\tan(\pi/4 - u/2)$$

So $y = \pi/4 - u/2$

Diff. w.r.t. u

$$\frac{dy}{du} = -1/2 \text{ —— Ans.}$$

16. $y = x^{\sin y}$

Sol. $y = x^{\sin y}$

Taking logarithm of both sides, we have

$\ln y = \ln x^{\sin y}$

or $\ln y = \sin y . \ln x$

Diff. w.r.t. x

$\frac{1}{y} . \frac{dy}{dx} = \sin y . \frac{1}{x} + \ln x . \cos y . \frac{dy}{dx}$

Multiplying both sides by xy.

$x \, dy/dx = y \sin y + xy \ln x . \cos y . dy/dx$

$x \, dy/dx - xy \cos y . \ln x . dy/dx = y \sin y$

$\frac{dy}{dx}(x - xy\cos y . \ln x) = y \sin y$

$\Rightarrow \frac{dy}{dx} = \frac{y \sin y}{x - xy\cos y . \ln x}$

17. $x^y = e^{x-y}$

Sol. Taking logarithm of both sides, we have

$\ln x^y = \ln e^{x-y}$

$y \ln x = (x-y) . \ln e$

or $y \ln x = x - y$

$y \ln x + y = x$

$y(\ln x + 1) = x$

$\Rightarrow y = \frac{x}{1 + \ln x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(1+\ln x) . 1 - x . \frac{1}{x}}{(1+\ln x)^2}$$

$$= \frac{1 + \ln x - 1}{(1+\ln x)^2}$$

$$\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$$

18.

$y^x + x^y = c$

Sol.- Given $y^x + x^y = c$

Sol. Let $u = y^x$ and $v = x^y$

Taking logarithm of both sides of the first equation

$\ln u = \ln y^x$

$\ln u = x \ln y$

Differentiating w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{y}\frac{dy}{dx} + \ln y$$

$$\frac{du}{dx} = u\left(\frac{x}{y}\cdot\frac{dy}{dx} + \ln y\right)$$

$$= y^x\left(\frac{x}{y}\frac{dy}{dx} + \ln y\right)$$

Now from $v = x^y$, taking logarithm, we get

$\ln v = \ln x^y$

or $\ln v = y \ln x$

Differentiating w.r.t. x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{y}{x} + \ln x\,\frac{dy}{dx}$$

$$\frac{dv}{dx} = v\left[\frac{y}{x} + \ln x \cdot \frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \ln x \cdot \frac{dy}{dx}\right]$$

The given equation is

$$u + v = c$$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (1) and (2) into (3), we have

$$y^x\left[\frac{x}{y}\frac{dy}{dx} + \ln y\right] + x^y\left[\frac{y}{x} + \ln x \cdot \frac{dy}{dx}\right] = 0$$

$$x y^{x-1} \, dy/dx + y^x \ln y + x^y \cdot y/x + x^y \cdot \ln x \cdot dy/dx = 0$$

or $$(x y^{x-1} + x^y \ln x)\frac{dy}{dx} + \left[y^x \ln y + x^y \cdot \frac{y}{x}\right] = 0$$

$$(x y^{x-1} + x^y \cdot \ln x)\cdot\frac{dy}{dx} = -(y^x \cdot \ln y + y \cdot x^{y-1})$$

or $\quad \dfrac{dy}{dx} = -\dfrac{y^x \ln y + y\,x^{y-1}}{xy^{x-1} + y^x \ln x}$

19. $\dfrac{x+y}{x-y} = x^2 + y^2$

Sol. Differentiating w.r.t. x, we have

$$\frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2} = 2x + 2yy'$$

or $\quad \dfrac{(x-y)+(x-y)y' - (x+y) + y'(x+y)}{(x-y)^2} = 2x+2yy'$

$$(x-y-x-y) + y'(x-\cancel{y}+x+\cancel{y}) = 2(x+yy')(x-y)^2$$

$$-2y + y'(2x) = 2(x+yy')(x-y)^2$$

$$-y + xy' = (x+yy')(x-y)^2$$

$$xy' - y = x(x-y)^2 + yy'(x-y)^2$$

$$xy' - yy'(x-y)^2 = x(x-y)^2 + y$$

$$y'(x-y(x-y)^2) = y + x(x-y)^2$$

$$\frac{dy}{dx} = \frac{y + x(x-y)^2}{x - y(x-y)^2}$$

20. $x + \text{arc}\sin y = xy$

Sol. $x + \sin^{-1} y = xy$

Differentiating (1) w.r.t. x, we have

$$1 + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = x\frac{dy}{dx} + y\cdot 1$$

$$\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} - x\frac{dy}{dx} = y-1$$

or $\quad \dfrac{dy}{dx}\left(\dfrac{1}{\sqrt{1-y^2}} - x\right) = y-1$

or $\quad \dfrac{dy}{dx}\left(\dfrac{1-x\sqrt{1-y^2}}{\sqrt{1-y^2}}\right) = y-1$

Therefore,

$$\frac{dy}{dx} = \frac{(y-1)\sqrt{1-y^2}}{1-x\sqrt{1-y^2}}$$

In Problems 21-30, find $f'(x)$:

21. $f(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

Sol. Here $f(x) = \ln\left(x + \sqrt{x^2-1}\right)$

Diff. w.r.t. x

$$f'(x) = \frac{1}{\left(x+\sqrt{x^2+1}\right)} \cdot \frac{d}{dx}\left(x + \sqrt{x^2-1}\right).$$

$$= \frac{1}{\left(x+\sqrt{x^2-1}\right)} \cdot \left(1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right)$$

$$= \frac{1}{\left(x+\sqrt{x^2-1}\right)} \cdot \left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{\left(x+\sqrt{x^2-1}\right)} \cdot \left[\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right]$$

$$= \frac{1}{\sqrt{x^2-1}}$$

22. $f(x) = \ln \dfrac{e^x}{1+e^x}$

Sol. $f(x) = \ln \dfrac{e^x}{1+e^x}$

$$= \ln e^x - \ln(1+e^x)$$

$$f(x) = x - \ln(1+e^x)$$

Diff. w.r.t. x

$$f'(x) = 1 - \frac{1}{1+e^x} \cdot e^x = 1 - \frac{e^x}{1+e^x}$$

$$= \frac{1+e^x - e^x}{1+e^x} = \frac{1}{1+e^x}$$

23. $f(x) = x^{\ln x}$

Sol. Taking ln of both sides, we get

$$\ln(f(x)) = \ln x^{\ln x}$$

$$\text{or } \ln f(x) = \ln x \cdot \ln x$$

Diff. w.r.t. x

$$\frac{1}{f(x)} \cdot f'(x) = \ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{x}$$

$$f'(x) = f(x)\left[\frac{\ln x}{x} + \frac{\ln x}{x}\right]$$

$$= x^{\ln x}\left[\frac{2\ln x}{x}\right]$$

$$= x^{\ln x - 1} \cdot (2\ln x)$$

24. $f(x) = \ln \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$

Sol. $f(x) = \ln\left(1+\sqrt{x}\right) - \ln\left(1-\sqrt{x}\right)$

Diff. w.r.t. x

$$f'(x) = \frac{1}{1+\sqrt{x}}\cdot\left(0+\frac{1}{2\sqrt{x}}\right) - \frac{1}{1-\sqrt{x}}\cdot\left(0-\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}(1+\sqrt{x})} + \frac{1}{2\sqrt{x}(1-\sqrt{x})}$$

$$= \frac{1}{2\sqrt{x}}\left[\frac{1}{1+\sqrt{x}} + \frac{1}{1-\sqrt{x}}\right]$$

$$= \frac{1}{2\sqrt{x}}\left[\frac{1-\sqrt{x}+1+\sqrt{x}}{(1+\sqrt{x})(1-\sqrt{x})}\right]$$

$$= \frac{1}{2\sqrt{x}}\cdot\frac{2}{(1-x)}$$

$$= \frac{1}{\sqrt{x}(1-x)}$$

25. $f(x) = e^{ax} \cos(b \operatorname{arc}\tan x)$

Sol. $f(x) = e^{ax} \cos(b\tan^{-1} x)$

Diff. w.r.t. x

$$f'(x) = e^{ax}\cdot -\sin(b\tan^{-1}x)\cdot b\cdot\frac{1}{1+x^2}) + \cos(b\tan^{-1}x)\cdot e^{ax}\cdot a$$

$$= e^{ax}\left[a\cos(b\tan^{-1}x) - \frac{b}{1+x^2}\cdot\sin(b\tan^{-1}x)\right]$$

$$= e^{ax}\left[\frac{a(1+x^2)\cdot\cos(b\tan^{-1}x) - b\sin(b\tan^{-1}x)}{(1+x^2)}\right]$$

26. $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\ln\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)}$

Sol.

Here $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\left[\ln\left(\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}\right) - \ln\left(\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}\right)\right]$

Diff. w.r.t. x

$$f'(x) = \frac{1}{\sqrt{b^2-a^2}}\left[\frac{1}{\sqrt{b+a}+\sqrt{b-a}\tan x/2}\cdot\sqrt{b-a}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2} - \frac{-\sqrt{b-a}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan x/2}\right]$$

$$f'(x) = \frac{1}{\sqrt{b^2-a^2}}\left[\frac{\frac{1}{2}\sqrt{b-a}\cdot\sec^2\frac{x}{2}}{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}} + \frac{\frac{1}{2}\sqrt{b-a}\cdot\sec^2\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan x/2}\right]$$ 16

$$= \frac{\sqrt{b-a}\sec^2\frac{x}{2}}{2\sqrt{(b-a)(b+a)}}\left[\frac{1}{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}} + \frac{1}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right]$$

$$= \frac{\sqrt{\cancel{b-a}}\sec^2\frac{x}{2}}{2\sqrt{\cancel{b-a}}\sqrt{b+a}}\cdot\left[\frac{\sqrt{b+a}-\sqrt{b-a}\cancel{\tan x/2}+\sqrt{b+a}+\sqrt{b-a}\cancel{\tan x/2}}{(\sqrt{b+a}+\sqrt{b-a}\tan x/2)(\sqrt{b+a}-\sqrt{b-a}\tan x/2)}\right]$$

$$= \frac{\sec^2\frac{x}{2}}{2\sqrt{\cancel{b+a}}}\cdot\frac{2\sqrt{\cancel{b+a}}}{(b+a)-(b-a)\cdot\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$$

$$= \frac{\sec^2\frac{x}{2}}{\frac{(b+a)\cos^2\frac{x}{2}-(b-a)\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} =$$

$$= \frac{\sec^2\frac{x}{2}\cdot\cos^2\frac{x}{2}}{b\cos^2\frac{x}{2}+a\cos^2\frac{x}{2}-b\sin^2\frac{x}{2}+a\sin^2\frac{x}{2}}$$

$$= \frac{1}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \frac{1}{a+b\cos x}$$

27. $f(x) = xa^x \sinh x$

Sol: $f(x) = x.a^x.\sinh x$

Diff. w.r.t. x

$$f'(x) = x\cdot a^x.\cosh x + x\sinh x.a^x.\ln a + 1.a^x.\sinh x$$
$$= a^x[x\cosh x + x\sinh x.\ln a + \sinh x]$$

28. $f(x) = -\dfrac{\cos x}{2\sin^2 x} + \dfrac{1}{2}\ln\tan\dfrac{x}{2}$

Sol.- Here $f(x) = -\frac{1}{2}\cdot\frac{\cos x}{\sin^2 x} + \frac{1}{2}\ln\tan\frac{x}{2}$

Diff. w.r.t. x

$$f'(x) = -\frac{1}{2}\left[\frac{\sin^2 x.(-\sin x)-\cos x.2\sin x\cos x}{\sin^4 x}\right]+\frac{1}{2}\cdot\frac{1}{\tan\frac{x}{2}}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}$$

$$= -\frac{1}{2}\left[\frac{-\sin^3 x-2\sin x\cos^2 x}{\sin^4 x}\right]+\frac{1}{4}\cdot\frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$$

$$= -\frac{1}{2}\left[\frac{-\sin^2 x-2\cos^2 x}{\sin^3 x}\right]+\frac{1}{4}\frac{\frac{1}{\cos^2 x/2}}{\frac{\sin x/2}{\cos x/2}}$$

$$= -\frac{1}{2}\left[\frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x}\right] + \frac{1}{(4\sin\frac{x}{2}\cos\frac{x}{2})}$$ 17

$$= -\frac{1}{2}\left[\frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x}\right] + \frac{1}{2(2\sin\frac{x}{2}\cos\frac{x}{2})}$$

$$= \frac{\sin^2 x + 2\cos^2 x}{2\sin^3 x} + \frac{1}{2\sin x}$$

$$= \frac{\sin^2 x + 2\cos^2 x + \sin^2 x}{2\sin^3 x} = \frac{2\sin^2 x + 2\cos^2 x}{2\sin^3 x}$$

$$= \frac{2(\sin^2 x + \cos^2 x)}{2\sin^3 x} = \frac{1}{\sin^3 x} = \operatorname{Cosec}^3 x$$

29. $f(x) = \operatorname{arcsec}\left(\csc x + \sqrt{x}\right)$

Sol:- Here $f(x) = \sec^{-1}(\operatorname{Cosec} x + \sqrt{x})$

Diff. w.r.t. x

$$f'(x) = \frac{1}{(\operatorname{Cosec} x + \sqrt{x})\sqrt{(\operatorname{Cosec} x + \sqrt{x})^2 - 1}} \cdot \frac{d}{dx}(\operatorname{Cosec} x + \sqrt{x})$$

$$= \frac{1}{(\operatorname{Cosec} x + \sqrt{x})\sqrt{(\operatorname{Cosec} x + \sqrt{x})^2 - 1}} \times \left(-\operatorname{Cosec} x . \operatorname{Cot} x + \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{(\operatorname{Cosec} x + \sqrt{x})\sqrt{\operatorname{Cosec}^2 x + x + 2\sqrt{x}\operatorname{Cosec} x - 1}} \cdot \left[\frac{-2\sqrt{x}\operatorname{Cosec} x \operatorname{Cot} x + 1}{2\sqrt{x}}\right]$$

$$= \frac{1 - 2\sqrt{x}\csc x \cot x}{2\sqrt{x}(\operatorname{Cosec} x + \sqrt{x})\sqrt{\operatorname{Cosec}^2 x + x + 2\sqrt{x}\operatorname{Cosec} x - 1}}$$

30. $f(x) = \left(1 + \frac{1}{x}\right)^{x^2}$

Sol. Taking ln of both sides, we have

$$\ln f(x) = \ln\left(1 + \frac{1}{x}\right)^{x^2}$$

$$\ln f(x) = x^2 \ln\left(\frac{x+1}{x}\right).$$

Diff. w.r.t. x

$$\frac{1}{f(x)} \cdot f'(x) = x^2 \cdot \frac{1}{\frac{x+1}{x}} \cdot \left[\frac{x \cdot 1 - (x+1) \cdot 1}{x^2}\right] + \ln\left(\frac{x+1}{x}\right) \cdot 2x$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{x}{x+1}\left[\frac{x - x - 1}{x^2}\right] + 2x\ln\left(\frac{x+1}{x}\right)$$

$$= \frac{-1}{x(x+1)} + 2x\ln\frac{(x+1)}{x}$$

$$\Rightarrow f'(x) = f(x)\left[2x\ln\left(\frac{x+1}{x}\right) - \frac{1}{x(x+1)}\right]$$ 18

$$= \left(1+\frac{1}{x}\right)^{x^2}\left[2x\ln\left(\frac{x+1}{x}\right) - \frac{1}{x(x+1)}\right]$$

Differentiate with respect to x each of the following (Problems 31-42)

31. $\arctan\left(\dfrac{1+2x}{2-x}\right)$

Sol.

Let $y = \tan^{-1}\left(\dfrac{1+2x}{2-x}\right)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1+2x}{2-x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1+2x}{2-x}\right)$$

$$= \frac{1}{1+\frac{(1+2x)^2}{(2-x)^2}}\cdot\frac{(2-x)\cdot 2-(1+2x)\cdot(-1)}{(2-x)^2}$$

$$= \frac{1}{\frac{(2-x)^2+(1+2x)^2}{(2-x)^2}}\cdot\frac{4-2x+1+2x}{(2-x)^2}$$

$$= \frac{\cancel{(2-x)^2}}{4-4x+x^2+1+4x+4x^2}\cdot\frac{5}{\cancel{(2-x)^2}} = \frac{5}{5+5x^2} = \frac{5}{5(1+x^2)} = \frac{1}{1+x^2}$$

32. $\ln(\arcsin e^x) + yx^2 = 1$

Sol. Given $\ln(\sin^{-1}e^x) + yx^2 = 1$

Diff. w.r.t. x

$$\frac{1}{\sin^{-1}(e^x)}\cdot\frac{d}{dx}(\sin^{-1}e^x) + \frac{d}{dx}(yx^2) = 0$$

$$\frac{1}{\sin^{-1}(e^x)}\cdot\frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x + y\cdot 2x + x^2\frac{dy}{dx} = 0$$

$$\frac{1}{\sin^{-1}(e^x)}\cdot\frac{e^x}{\sqrt{1-e^{2x}}} + 2xy + x^2\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} = -2xy - \frac{e^x}{\sin^{-1}(e^x)\sqrt{1-e^{2x}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2}\left[-2xy - \frac{e^x}{\sin^{-1}(e^x)\sqrt{1-e^{2x}}}\right] = -\frac{2y}{x} - \frac{e^x}{x^2\sin^{-1}(e^x)\sqrt{1-e^{2x}}}$$

33. $y = (\arcsin x^2)^{\pi}$

Sol. Let $y = (\sin^{-1} x^2)^{\pi}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \pi(\sin^{-1}x^2)^{\pi-1} \cdot \frac{d}{dx}(\sin^{-1}x^2)$$

$$= \pi(\sin^{-1}x^2)^{\pi-1} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$= \pi \cdot (\sin^{-1}x^2)^{\pi-1} \cdot \frac{2x}{\sqrt{1-x^4}}$$

$$= \frac{2x\pi(\sin^{-1}x^2)^{\pi-1}}{\sqrt{1-x^4}}$$

34. $\arctan\left(\frac{y}{x}\right) + yx^2 = 1$

Sol. Given $\tan^{-1}(y/x) + yx^2 = 1$

Diff. w.r.t. x

$$\frac{1}{1+(y/x)^2} \cdot \left(\frac{x\,dy/dx - y \cdot 1}{x^2}\right) + y \cdot 2x + x^2\frac{dy}{dx} = 0$$

$$\frac{1}{1+y^2/x^2} \cdot \left(\frac{x\,dy/dx - y}{x^2}\right) + 2xy + x^2\frac{dy}{dx} = 0$$

$$\frac{1}{\frac{x^2+y^2}{x^2}} \cdot \left(\frac{x\frac{dy}{dx} - y}{x^2}\right) + 2xy + x^2\frac{dy}{dx} = 0$$

$$\frac{x^2}{x^2+y^2} \cdot \frac{x\frac{dy}{dx} - y}{x^2} + 2xy + x^2\frac{dy}{dx} = 0$$

$$\frac{x\,dy/dx - y}{x^2+y^2} + x^2\frac{dy}{dx} = -2xy$$

or $x\frac{dy}{dx} - y + x^2(x^2+y^2)\frac{dy}{dx} = -2xy(x^2+y^2)$

$dy/dx\,(x + x^4 + x^2y^2) = y - 2xy(x^2+y^2)$

$\Rightarrow dy/dx = \dfrac{y(1-2x(x^2+y^2))}{x(1+x^3+xy^2)}$ — Ans

35. $y = \dfrac{1-\cosh x}{1+\cosh x}$

Sol.

Given $y = \dfrac{1-\cosh x}{1+\cosh x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(1+\cosh x)\cdot\frac{d}{dx}(1-\cosh x) - (1-\cosh x)\cdot\frac{d}{dx}(1+\cosh x)}{(1+\cosh x)^2}$$

$$= \frac{(1+\cosh x)(-\sinh x) - (1-\cosh x)\cdot\sinh x}{(1+\cosh x)^2}$$

$$= \frac{-\sinh x - \cosh x \cdot \sinh x - \sinh x + \cosh x\cdot\sinh x}{(1+\cosh x)^2} = \frac{-2\sinh x}{(1+\cosh x)^2}$$

$$= \frac{-2\cdot 2\sinh x/2\,\cosh x/2}{(2\cosh^2 x/2)^2} = \frac{-4\sinh x/2\,\cosh x/2}{4\cosh^4 x/2} = -\frac{\sinh x/2}{\cosh x/2}\cdot\frac{1}{\cosh^2 x/2} = -\tanh\frac{x}{2}\,\operatorname{sech}^2\frac{x}{2}$$

36. $y = \ln(\tanh 2x)$

Sol.- $y = \ln(\tanh 2x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\tanh 2x} \cdot \frac{d}{dx}(\tanh 2x)$$

$$= \frac{1}{\tanh 2x} \cdot \operatorname{sech}^2 2x \cdot 2$$

$$= \coth 2x \cdot \operatorname{sech}^2 2x \cdot 2$$

$$= \frac{\cosh 2x}{\sinh 2x} \cdot \frac{1}{\cosh^2 2x} \cdot 2$$

$$= \frac{2}{\sinh 2x \cdot \cosh 2x} = \frac{4}{2\sinh 2x \cdot \cosh 2x}$$

$$= \frac{4}{\sinh 4x} = 4\operatorname{cosech} 4x$$

37. $\log_{10}\left(\dfrac{x+1}{x}\right)$

Sol. $y = \log_{10}\left(\dfrac{x+1}{x}\right)$

$$y = \frac{\log_e\left(\frac{x+1}{x}\right)}{\log_e 10} = \frac{\ln\left(\frac{x+1}{x}\right)}{\ln 10}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{1}{\frac{x+1}{x}} \cdot \left[\frac{x \cdot 1 - (x+1) \cdot 1}{x^2}\right]$$

$$= \frac{x}{(x+1) \cdot \ln 10} \cdot \left(\frac{x - x - 1}{x^2}\right) = \frac{-x}{(x+1) \cdot \ln 10 \cdot x^2}$$

$$= \frac{-1}{(x+1) \cdot \ln 10 \cdot x} \text{ — Ans.}$$

38. $\arccos\sqrt{1-x^2}$

Sol. Let $y = \cos^{-1}\sqrt{1-x^2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= \frac{-1}{\sqrt{1-(1-x^2)}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{x}{\sqrt{1-1+x^2} \cdot \sqrt{1-x^2}}$$

$$= \frac{x}{\sqrt{x^2}} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{x}{|x|} \cdot \frac{1}{\sqrt{1-x^2}}$$

39. $\text{arc Sec}(\sinh x)$

Sol:- Let $y = \sec^{-1}(\sinh x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\sinh x\sqrt{\sinh^2 x-1}} \cdot \frac{d}{dx}(\sinh x)$$

$$= \frac{1}{\sinh x\sqrt{\sinh^2 x-1}} \cdot \cosh x$$

$$= \frac{\coth x}{\sqrt{\sinh^2 x-1}}$$

40. $\text{arcSin}(\text{arcCot}\ln x)$

Sol. Let $y = \sin^{-1}(\cot^{-1}\ln x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\cot^{-1}\ln x)^2}} \cdot \frac{d}{dx}(\cot^{-1}\ln x)$$

$$= \frac{1}{\sqrt{1-(\cot^{-1}\ln x)^2}} \cdot \frac{-1}{1+(\ln x)^2} \cdot \frac{1}{x}$$

$$= \frac{-1}{x(1+\ln^2 x)\sqrt{1-(\cot^{-1}\ln x)^2}}$$

41. $\cosh^{-1}(1+x^2)$

Sol:- Let $y = \cosh^{-1}(1+x^2)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)^2-1}} \cdot \frac{d}{dx}(1+x^2)$$

$$= \frac{1}{\sqrt{\cancel{1}+2x^2+x^4-\cancel{1}}} \cdot 2x = \frac{2x}{\sqrt{2x^2+x^4}} = \frac{2x}{|x|\sqrt{2+x^2}}$$

42. $\sinh^{-1}(\tanh x)$

Sol:- Let $y = \sinh^{-1}(\tanh x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\tanh^2 x}} \cdot \frac{d}{dx}(\tanh x) = \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}} \text{ — Ans.}$$

Differentiate (logarithmically) with respect to x **(Problems 43-47)**

43. $y = \sqrt[3]{\dfrac{x\,(x^2+1)}{(x-1)^2}}$

Sol.
Here $y = \left[\dfrac{x(x^2+1)}{(x-1)^2}\right]^{1/3}$

taking ln on both sides:

$$\ln y = \ln\left[\frac{x(x^2+1)}{(x-1)^2}\right]^{1/3}$$

$$= \frac{1}{3}\ln\left[\frac{x(x^2+1)}{(x-1)^2}\right]$$

$$= \frac{1}{3}\left[\ln x + \ln(x^2+1) - \ln(x-1)^2\right]$$

$$\ln y = \frac{1}{3}\left[\ln x + \ln(x^2+1) - 2\ln(x-1)\right]$$

Diff. w.r.t. x

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x^2+1}\cdot 2x - 2\cdot\frac{1}{x-1}\cdot 1\right] = \frac{1}{3}\left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{2}{x-1}\right]$$

$$= \frac{1}{3}\left[\frac{(x^2+1)(x-1) + 2x\cdot x(x-1) - 2x(x^2+1)}{x(x^2+1)(x-1)}\right] = \frac{1}{3}\left[\frac{x^3 - x^2 + x - 1 + 2x^3 - 2x^2 - 2x^3 - 2x}{x(x^2+1)(x-1)}\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{3}\left[\frac{x^3-3x^2-x-1}{x(x^2+1)(x-1)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{3}\left[\frac{x^3-3x^2-x-1}{x(x^2+1)(x-1)}\right] = \frac{x^{1/3}\cdot(x^2+1)^{1/3}}{3(x-1)^{2/3}}\cdot\frac{x^3-3x^2-x-1}{x(x^2+1)(x-1)}$$

$$= \frac{x^3-3x^2-x-1}{3x^{1-1/3}\cdot(x-1)^{2/3+1}\cdot(x^2+1)^{1-1/3}} = \frac{x^3-3x^2-x-1}{3x^{2/3}\,(x-1)^{5/3}\,(x^2+1)^{2/3}} \text{ Ans.}$$

44. $y = \dfrac{\sqrt{x}\;(1-2x)^{2/3}}{(2-3x)^{3/4}\;(3-4x)^{4/3}}$

Sol. Taking logarithm of both sides, we get

$$\ln y = \ln\left[\frac{\sqrt{x}\,(1-2x)^{2/3}}{(2-3x)^{3/4}\cdot(3-4x)^{4/3}}\right]$$

$$= \ln\left(\sqrt{x}\,(1-2x)^{2/3}\right) - \ln\left((2-3x)^{3/4}\cdot(3-4x)^{4/3}\right)$$

$$= \left[\ln\sqrt{x} + \ln(1-2x)^{2/3}\right] - \left[\ln(2-3x)^{3/4} + \ln(3-4x)^{4/3}\right]$$

$$= \ln x^{1/2} + \ln(1-2x)^{2/3} - \ln(2-3x)^{3/4} - \ln(3-4x)^{4/3}$$

$$\ln y = \frac{1}{2}\ln x + \frac{2}{3}\ln(1-2x) - \frac{3}{4}\ln(2-3x) - \frac{4}{3}\ln(3-4x)$$

Diff. w.r.t. x

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x} + \frac{2}{3}\cdot\frac{1}{1-2x}\cdot(-2) - \frac{3}{4}\cdot\frac{1}{2-3x}(-3) - \frac{4}{3}\cdot\frac{1}{3-4x}(-4)$$

$$= \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{3(3-4x)}$$

$$= \frac{1}{2x} + \frac{9}{4(2-3x)} + \frac{-4}{3(1-2x)} + \frac{16}{3(3-4x)}$$

$$= \frac{2(2-3x)+9x}{4x(2-3x)} + \frac{-4(3-4x)+16(1-2x)}{3(1-2x)(3-4x)}$$

$$= \frac{4-6x+9x}{4x(2-3x)} + \frac{-12+16x+16-32x}{3(1-2x)(3-4x)}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{4+3x}{4x(2-3x)} + \frac{4-16x}{3(1-2x)(3-4x)}$$

$$\frac{dy}{dx} = y\left[\frac{4+3x}{4x(2-3x)} + \frac{4-16x}{3(1-2x)(3-4x)}\right]$$

$$= \frac{\sqrt{x}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}\left[\frac{4+3x}{4x(2-3x)} + \frac{4(1-4x)}{3(1-2x)(3-4x)}\right] \text{ — Ans.}$$

45. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$ ——— (1)

Sol. Let $u = (\tan x)^{\cot x}$

taking ln on both sides

$\ln u = \ln(\tan x)^{\cot x}$

$\ln u = \cot x \ln(\tan x)$

Differentiating, w.r.t. x, we have

$$\frac{1}{u}\cdot\frac{du}{dx} = \cot x\cdot\frac{1}{\tan x}\cdot\sec^2 x + \ln(\tan x)\cdot -\operatorname{cosec}^2 x$$

$$\frac{du}{dx} = u\left[\cot x\cdot\cot x\cdot\sec^2 x - \ln(\tan x)\cdot\operatorname{cosec}^2 x\right]$$

$$= (\tan x)^{\cot x}\left[\cot^2 x\cdot\sec^2 x - \ln(\tan x)\cdot\operatorname{cosec}^2 x\right]$$

$$= (\tan x)^{\cot x}\left[\frac{\cos^2 x}{\sin^2 x}\cdot\frac{1}{\cos^2 x} - \ln(\tan x)\cdot\operatorname{cosec}^2 x\right]$$

$$\frac{du}{dx} = (\tan x)^{\cot x}\left[\operatorname{cosec}^2 x - \ln(\tan x)\cdot\operatorname{cosec}^2 x\right]$$

& $v = (\cot x)^{\tan x}$

$\ln v = \ln(\cot x)^{\tan x}$

$\ln v = \tan x\cdot\ln(\cot x)$

Diff. w.r.t. x

$$\frac{1}{v}\cdot\frac{dv}{dx} = \tan x\cdot\frac{1}{\cot x}\cdot -\operatorname{cosec}^2 x + \ln(\cot x)\cdot\sec^2 x$$

$$\frac{dv}{dx} = v\left[-\tan x\cdot\tan x\cdot\operatorname{cosec}^2 x + \ln(\cot x)\cdot\sec^2 x\right]$$

$$= v\left[-\tan^2 x\cdot\operatorname{cosec}^2 x + \ln(\cot x)\cdot\sec^2 x\right]$$

$$= (\cot x)^{\tan x}\left[-\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\sin^2 x} + \ln(\cot x)\cdot\sec^2 x\right]$$

$$dv/dx = (\cot x)^{\tan x}\left[-\sec^2 x + \ln(\cot x)\cdot\sec^2 x\right]$$

Now Eq. (1) is

$y = u + v$

Diff. w.r.t. x

$\frac{dy}{dx} = \frac{du}{dx} + dv/dx$

Putting values

$\frac{dy}{dx} = (\tan x)^{\cot x}\left[\operatorname{cosec}^2 x - \ln(\tan x).\operatorname{cosec}^2 x\right] + (\cot x)^{\tan x}\left[-\sec^2 x + \ln(\cot x).\sec^2 x\right]$ — Ans

46. $y = x^x \cdot e^x \sin x \,.\, \ln x$

Sol. Taking logarithm of both sides, we get

$\ln y = \ln\left(x^x \cdot e^x \cdot \sin x . \ln x\right)$

$= \ln x^x + \ln e^x + \ln \sin x + \ln(\ln x)$

$= x \ln x + x \ln e + \ln \sin x + \ln(\ln x)$

$\ln y = x.\ln x + x + \ln \sin x + \ln(\ln x)$

Diff. w.r.t. x

$\frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + \ln x.1 + 1 + \frac{1}{\sin x}.\cos x + \frac{1}{\ln x}.\frac{1}{x}$

$\frac{dy}{dx} = y\left[1 + \ln x + 1 + \cot x + \frac{1}{x \ln x}\right]$

$= (x^x . e^x . \sin x . \ln x)\left[2 + \ln x + \cot x + \frac{1}{x \ln x}\right]$ —— Ans

In Problems 48 – 60, find $\frac{dy}{dx}$:

47. $y = \dfrac{(x+2)^2}{(x+1)(x^2+3)^3}$

Sol. Taking ln of both sides, we have

$\ln y = \ln\left[\frac{(x+2)^2}{(x+1)(x^2+3)^3}\right] = \ln(x+2)^2 - \ln(x+1) - \ln(x^2+3)^3$

$\ln y = 2\ln(x+2) - \ln(x+1) - 3\ln(x^2+3)$

Diff. w.r.t. x

$\frac{1}{y}.\frac{dy}{dx} = 2.\frac{1}{x+2} - \frac{1}{x+1} - 3.\frac{1}{x^2+3}.2x$

$\frac{dy}{dx} = y\left[\frac{2}{x+2} - \frac{1}{x+1} - \frac{6x}{x^2+3}\right] = y\left[\frac{2(x+1)(x^2+3) - (x+2)(x^2+3) - 6x(x+1)(x+2)}{(x+1)(x+2)(x^2+3)}\right]$

$= y\left[\frac{2x^3+6x+2x^2+6-x^3-3x-2x^2-6-6x(x^2+3x+2)}{(x+1)(x+2)(x^2+3)}\right]$

$= y\left[\frac{x^3+3x-6x^3-18x^2-12x}{(x+1)(x+2)(x^2+3)}\right] = \frac{(x+2)^2}{(x+1)(x^2+3)^3}\left[\frac{-5x^3-18x^2-9x}{(x+1)(x+2)(x^2+3)}\right]$

48. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Sol. Differentiating, both sides w.r.t. x, we get

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$

$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}.\frac{dy}{dx} = 0$

$\frac{1}{\sqrt{y}}.\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

or $\dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$

49. $x^3 + y^3 - 3\,a\,x\,y = 0$

Sol. Differentiating w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left[x\frac{dy}{dx} + y\,.\,1\right] = 0$$

or $\quad x^2 + y^2\dfrac{dy}{dx} - a\left[x\dfrac{dy}{dx} + y\,.\,1\right] = 0$

$$x^2 + y^2\,dy/dx - ax\,.\,dy/dx - ay = 0$$

$$\frac{dy}{dx}(y^2 - ax) = ay - x^2$$

or $\quad \dfrac{dy}{dx} = \dfrac{a\,y - x^2}{y^2 - a\,x}$

50. $y - \cos(x+y) = 0$

Sol. It can be rewritten as

$$y = \cos(x+y)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(x+y)\left[1 + \frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = -\sin(x+y) - \sin(x+y).\,dy/dx$$

$$\frac{dy}{dx} + \sin(x+y).\frac{dy}{dx} = -\sin(x+y).$$

$$\frac{dy}{dx}\,(1 + \sin(x+y)) = -\sin(x+y).$$

or $\quad \dfrac{dy}{dx} = -\dfrac{\sin(x+y)}{1+\sin(x+y)}$

51. $\arctan(x+y) = \arcsin(e^y + x)$

Sol. $\tan^{-1}(x+y) = \sin^{-1}(e^y + x)$

Diff. w.r.t. x

$$\frac{1}{1+(x+y)^2}\cdot\left(1 + dy/dx\right) = \frac{1}{\sqrt{1-(e^y+x)^2}}\cdot\left(e^y\cdot\frac{dy}{dx} + 1\right)$$

$$\frac{1}{1+(x+y)^2} + \frac{1}{1+(x+y)^2}\cdot\frac{dy}{dx} = \frac{1}{\sqrt{1-(e^y+x)^2}}\cdot e^y\frac{dy}{dx} + \frac{1}{\sqrt{1-(e^y+x)^2}}$$

$$\frac{1}{1+(x+y)^2}\cdot\frac{dy}{dx} - \frac{e^y}{\sqrt{1-(e^y+x)^2}}\cdot\frac{dy}{dx} = \frac{1}{\sqrt{1-(e^y+x)^2}} - \frac{1}{1+(x+y)^2}$$

$$\frac{dy}{dx}\left[\frac{1}{1+(x+y)^2} - \frac{e^y}{\sqrt{1-(e^y+x)^2}}\right] = \frac{1+(x+y)^2 - \sqrt{1-(e^y+x)^2}}{\sqrt{1-(e^y+x)^2}\,(1+(x+y)^2)}$$

$$\frac{dy}{dx}\left[\frac{\sqrt{1-(e^y+x)^2}-e^y\left[1+(x+y)^2\right]}{\left[1+(x+y)^2\right]\sqrt{1-(e^y+x)^2}}\right]=\frac{\left[1+(x+y)^2-\sqrt{1-(e^y+x)^2}\right]}{\sqrt{1-(e^y+x)^2}\left[1+(x+y)^2\right]}$$ 2.6

$$\frac{dy}{dx}\left[\sqrt{1-(e^y+x)^2}-e^y\left(1+(x+y)^2\right)\right] = 1+(x+y)^2-\sqrt{1-(e^y+x)^2}$$

or $$\frac{dy}{dx}=\frac{1+(x+y)^2-\sqrt{1-(e^y+x)^2}}{\sqrt{1-(e^y+x)^2}-e^y(1+(x+y)^2)}.$$

52. $x = a(t-\sin t),\ y = a(1-\cos t)$

Sol. Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = a(1-\cos t)$$

and $$\frac{dy}{dt} = a(\sin t) = a\sin t$$

Now $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{a\sin t}{a(1-\cos t)}$$

$$= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \cot\frac{t}{2}.$$

53. $x = a\cos^3 t,\ y = b\sin^3 t$

Sol. $x = a\cos^3 t$ & $y = b\sin^3 t$

Diff. w.r.t. t

$$\frac{dx}{dt} = a.3\cos^2 t.(-\sin t)$$
$$= -3a\cos^2 t.\sin t$$

& as $y = b\sin^3 t$

Diff. w.r.t. t

$$\frac{dy}{dt} = b(3\sin^2 t.\cos t)$$
$$= 3b\sin^2 t\cos t$$

Now $$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$
$$= \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t}$$
$$= -\frac{b}{a}.\frac{\sin t}{\cos t} = -\frac{b}{a}\tan t$$

54. $x = \dfrac{3at}{1+t^2}$, $y = \dfrac{3at^2}{1+t^2}$

Sol. Diff. w.r.t. t

$$\frac{dx}{dt} = 3a\left[\frac{(1+t^2).1 - t.(2t)}{(1+t^2)^2}\right]$$

$$= 3a\left[\frac{1+t^2-2t^2}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{3a(1-t^2)}{(1+t^2)^2}$$

Now $$\frac{dy}{dt} = 3a\left[\frac{(1+t^2).2t - t^2(2t)}{(1+t^2)^2}\right]$$

$$= 3a\left[\frac{2t+2t^3-2t^3}{(1+t^2)^2}\right]$$

$$= \frac{3a(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at}{(1+t^2)^2}$$

Now $$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{6at}{(1+t^2)^2}.\frac{(1+t^2)^2}{3a(1-t^2)}$$

$$= \frac{2t}{(1-t^2)} \text{ —— Ans.}$$

55. $y = (\sec x^3 + \operatorname{arcsec} x)^2$

Sol. $y = (\sec x^3 + \sec^{-1} x)^2$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2(\sec x^3 + \sec^{-1}x).\frac{d}{dx}(\sec x^3 + \sec^{-1}x)$$

$$= 2(\sec x^3 + \sec^{-1}x).\left[\sec x^3.\tan x^3.3x^2 + \frac{1}{x\sqrt{x^2-1}}\right]$$

$$= 2(\sec x^3 + \sec^{-1}x)\left[3\sec x^3.\tan x^3.x^2 + \frac{1}{x\sqrt{x^2-1}}\right] \text{ —— Ans.}$$

56. $y = \exp\left(\operatorname{arccsc}\left(\dfrac{1}{x}\right)\right)$

Sol. $y = e^{\csc^{-1}(1/x)}$

Diff. w.r.t. x

$$\frac{dy}{dx} = e^{\csc^{-1}(1/x)}.\frac{-1}{\frac{1}{x}\sqrt{(\frac{1}{x})^2-1}}.\frac{-1}{x^2}$$

$$= e^{\csc^{-1}(1/x)}\frac{1}{x\sqrt{\frac{1}{x^2}-1}} = e^{\csc^{-1}(\frac{1}{x})}.\frac{1}{x\sqrt{\frac{1-x^2}{x^2}}}$$

$$\frac{dy}{dx} = e^{\cos^{-1}(1/x)} \cdot \frac{1}{\frac{x\sqrt{1-x^2}}{x}}$$

$$\frac{dy}{dx} = e^{\cos^{-1}\frac{1}{x}} \cdot \frac{1}{\sqrt{1-x^2}} \quad \text{—— Ans.}$$

57. $y = \arcsin(\ln x) - \ln(\arctan x)$

Sol. $y = \sin^{-1}(\ln x) - \ln(\tan^{-1} x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{1}{x} - \frac{1}{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{x\sqrt{1-(\ln x)^2}} - \frac{1}{(1+x^2)\tan^{-1} x}$$

58. $y \arcsin x - x \arctan y = 1$

Sol. $y \sin^{-1} x - x \tan^{-1} y = 1$

Diff. w.r.t. x

$$y \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot \frac{dy}{dx} - x \cdot \frac{1}{1+y^2} \cdot \frac{dy}{dx} - \tan^{-1} y = 0$$

$$\frac{dy}{dx}\left(\sin^{-1} x - \frac{x}{1+y^2}\right) + \frac{y}{\sqrt{1-x^2}} - \tan^{-1} y = 0$$

$$\frac{dy}{dx} \cdot \left(\frac{(1+y^2)\sin^{-1} x - x}{(1+y^2)}\right) = -\frac{y}{\sqrt{1-x^2}} + \tan^{-1} y$$

$$\frac{dy}{dx} \cdot \left(\frac{(1+y^2)\sin^{-1} x - x}{1+y^2}\right) = \frac{-y + \sqrt{1-x^2}\tan^{-1} y}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1+y^2}{(1+y^2)\sin^{-1} x - x} \times \frac{\left[-y + \sqrt{1-x^2} \cdot \tan^{-1} y\right]}{\sqrt{1-x^2}}$$

$$= \frac{(1+y^2)\left(\sqrt{1-x^2}\tan^{-1} y - y\right)}{\sqrt{1-x^2}\left[(1+y^2)\sin^{-1} x - x\right]} \quad \text{—— Ans.}$$

59. $\arcsin(\ln xy) = x + y^2$

Sol. $\sin^{-1}(\ln xy) = x + y^2$

Diff. w.r.t. x

$$\frac{1}{\sqrt{1-(\ln xy)^2}} \cdot \frac{d}{dx}(\ln xy) = 1 + 2y \cdot \frac{dy}{dx}$$

$$\frac{1}{\sqrt{1-(\ln xy)^2}} \cdot \frac{1}{xy} \cdot \left[x\frac{dy}{dx} + y \cdot 1\right] = 1 + 2y \cdot \frac{dy}{dx}$$

$$x\frac{dy}{dx} + y = xy\sqrt{1-(\ln xy)^2}\left[1 + 2y \cdot \frac{dy}{dx}\right]$$

$$x\frac{dy}{dx} + y = xy\sqrt{1-(\ln xy)^2} + 2xy^2\sqrt{1-(\ln xy)^2} \cdot \frac{dy}{dx}$$

$$x\frac{dy}{dx} - 2xy^2\sqrt{1-(\ln xy)^2} \cdot \frac{dy}{dx} = xy\sqrt{1-(\ln xy)^2} - y$$

$$x\frac{dy}{dx}\left(1-2y^2\sqrt{1-(\ln xy)^2}\right) = y\left(x\sqrt{1-(\ln xy)^2}-1\right)$$ (29)

$$\frac{dy}{dx} = \frac{y\left(x\sqrt{1-(\ln xy)^2}-1\right)}{x\left(1-2y^2\sqrt{1-(\ln xy)^2}\right)} = \frac{y\left(x\sqrt{1-\ln^2 xy}-1\right)}{x\left(1-2y^2\sqrt{1-\ln^2 xy}\right)}$$

60. $\operatorname{arcsec}(x^2+y)-e^x = \dfrac{1}{x+y}$

Sol. $\sec^{-1}(x^2+y)-e^x = \dfrac{1}{x+y}$

Diff. w.r.t. x

$$\frac{1}{(x^2+y)\sqrt{(x^2+y)^2-1}}\cdot\frac{d}{dx}(x^2+y)-e^x = \frac{d}{dx}(x+y)^{-1}$$

$$\frac{1}{(x^2+y)\sqrt{(x^2+y)^2-1}}\cdot\left(2x+\frac{dy}{dx}\right)-e^x = -(x+y)^{-2}\cdot\left(1+\frac{dy}{dx}\right)$$

$$\frac{2x+\frac{dy}{dx}}{(x^2+y)\sqrt{(x^2+y)^2-1}}-e^x = \frac{-1-\frac{dy}{dx}}{(x+y)^2}$$

$$\frac{2x+\frac{dy}{dx}-e^x(x^2+y)\sqrt{(x^2+y)^2-1}}{(x^2+y)\sqrt{(x^2+y)^2-1}} = \frac{-1-\frac{dy}{dx}}{(x+y)^2}$$

$$(x+y)^2\left(2x+\frac{dy}{dx}\right)-(x+y)^2\cdot e^x(x^2+y)\sqrt{(x^2+y)^2-1} = \left(-1-\frac{dy}{dx}\right)(x^2+y)\sqrt{(x^2+y)^2-1}$$

$$2x(x+y)^2+(x+y)^2\cdot\frac{dy}{dx}-(x+y)^2\cdot e^x\cdot(x^2+y)\sqrt{(x^2+y)^2-1} = -(x^2+y)\sqrt{(x^2+y)^2-1}-\frac{dy}{dx}(x^2+y)\sqrt{(x^2+y)^2-1}$$

$$(x+y)^2\cdot\frac{dy}{dx}+\frac{dy}{dx}(x^2+y)\sqrt{(x^2+y)^2-1} = (x+y)^2\cdot e^x(x^2+y)\sqrt{(x^2+y)^2-1}-(x^2+y)\sqrt{(x^2+y)^2-1}-2x(x+y)^2$$

$$\frac{dy}{dx}\left((x+y)^2+(x^2+y)\sqrt{(x^2+y)^2-1}\right) = e^x(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}-(x^2+y)\sqrt{(x^2+y)^2-1}-2x(x+y)^2$$

$$\frac{dy}{dx} = \frac{e^x(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}-(x^2+y)\sqrt{(x^2+y)^2-1}-2x(x+y)^2}{(x+y)^2+(x^2+y)\sqrt{(x^2+y)^2-1}}$$ Ans

 BSc Mathematics
Calculus- Chapter 02

2.1-1

Ch-2 (Derivatives)

Derivative of a function:

Let f be a function from R to R then derivative of f at x is defined by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

The above limit will exist only left & right hand limits

$$Lf'(x) = \lim_{h\to 0-0} \frac{f(x+h)-f(x)}{h}$$

& $Rf'(x) = \lim_{h\to 0+0} \dfrac{f(x+h)-f(x)}{h}$ exist & are equal

In this case we say f is differentiable or derivable at x.

Note The derivative of a function f at a pt. $x=a$ is defined as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

<u>Theorem</u> If f is differentiable at a pt. $x = a \in D_f$ then f is Continuous at $x = a$.

<u>Proof:</u> Given that f is differentiable at $x = a$

To show that f is Continuous at $x = a$ we have to show that

$$\lim_{x \to a} f(x) = f(a)$$

As f is differentiable at $x = a$

So $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

Consider $\lim_{x \to a} \left(f(x) - f(a)\right) = \lim_{x \to a} \left[\dfrac{f(x) - f(a)}{x - a} \times (x - a) \right]$

$$= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \to a} (x - a) \right)$$

$$= f'(a) \cdot (a - a)$$

$$= f'(a) \cdot (0)$$

$$\Rightarrow \lim_{x \to a} \left(f(x) - f(a)\right) = 0$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a)$$

Which shows that f is Continuous at $x = a$.

<u>Note</u> The Converse of this theorem does not hold. i.e., a Continuous function may not be differentiable. We give an example to prove it.

2-1-3

Example Let $f: R \to R$ be defined by 3

$$f(x) = |x|$$

then prove that f is continuous at $x=0$ but is not differentiable at $x=0$

Sol:- Given function is

$$f(x) = |x|$$

Here $f(0) = 0$

$$f(0-0) = \lim_{x \to 0-0} |x| = \lim_{x \to 0-0} (-x) = 0$$

& $$f(0+0) = \lim_{x \to 0+0} |x| = \lim_{x \to 0+0} (x) = 0$$

Since $f(0-0) = f(0+0) = f(0)$

So f is continuous at $x=0$

Now we check the derivability of f

As $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

So $$f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h|-0}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

Now $$Lf'(0) = \lim_{h \to 0-0} \frac{|h|}{h} = \lim_{h \to 0-0} \frac{-h}{h} = -1$$

& $$Rf'(0) = \lim_{h \to 0+0} \frac{|h|}{h} = \lim_{h \to 0+0} \frac{h}{h} = 1$$

Since $Lf'(0) \neq Rf'(0)$. So f is not derivable at $x=0$

(Exercise No. 2.1)

Q1 Show that the function $f(x) = |x| + |x-1|$ is continuous for every value of x but is not differentiable at $x = 0$ & $x = 1$

Sol:- Given function is

$$f(x) = |x| + |x-1|$$

First we discuss the continuity of f

Let x_0 be an arbitrary real no. then

$$f(x_0) = |x_0| + |x_0 - 1|$$

Now $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} (|x| + |x-1|)$

$$= |x_0| + |x_0 - 1|$$

Since $\lim\limits_{x \to x_0} f(x) = f(x_0)$

So f is continuous at $x = x_0$. But x_0 is any real no. So f is continuous for every real value of x.

Now we will show that f is not differentiable at $x = 0$ & $x = 1$

Since $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

So $f'(0) = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h}$

Now $Lf'(0) = \lim\limits_{h \to 0-0} \dfrac{|h| + |h-1| - |-1|}{h}$

$$= \lim_{h \to 0-0} \frac{-h - (h-1) - 1}{h}$$

$$= \lim_{h \to 0-0} \frac{-h - h + 1 - 1}{h}$$

$$= \lim_{h \to 0-0} \frac{-2h}{h}$$

$$= \lim_{h \to 0-0} (-2)$$

$$Lf'(0) = -2$$

& $$Rf'(0) = \lim_{h \to 0+0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0+0} \frac{|h| + |h-1| - |-1|}{h}$$

$$= \lim_{h \to 0+0} \frac{h - (h-1) - 1}{h}$$

$$= \lim_{h \to 0+0} \frac{h - h + 1 - 1}{h}$$

$$= \lim_{h \to 0+0} \frac{0}{h}$$

$$= \lim_{h \to 0+0} (0)$$

$$Rf'(0) = 0$$

Since $Lf'(0) \neq Rf'(0)$

So f is not differentiable at $x = 0$

Now $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$Lf'(1) = \lim_{h \to 0-0} \frac{(|1+h| + |1+h-1|) - (|1| + |1-1|)}{h}$$

$$= \lim_{h \to 0-0} \frac{1 + h - h - 1}{h}$$

$$= \lim_{h \to 0-0} (0)$$

So $Lf'(1) = 0$

& $Rf'(1) = \lim\limits_{h\to 0+0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{(|1+h|+|1+h-1|)-(|1|+|1-1|)}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{1+h+h-1}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{2h}{h}$

$= \lim\limits_{h\to 0+0} (2)$

$Rf'(1) = 2$

Since $Lf'(1) \neq Rf'(1)$

So f is not derivable at $x=1$

<u>Q2</u> Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2x-1 & 1 < x \le 2 \end{cases}$

Discuss the Continuity & differentiability of f at $x=1$

<u>Sol.</u> Given eq. is

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2x-1 & 1 < x \le 2 \end{cases}$$

First we will discuss the Continuity of f

Here $f(1) = 1$

& $f(1-0) = \lim\limits_{x\to 1-0} f(x)$

$= \lim\limits_{x\to 1-0} (x)$

7

$$f(1-0) = 1$$

& $$f(1+0) = \lim_{x\to 1+0} f(x)$$

$$= \lim_{x\to 1+0} (2x-1)$$

$$= 2(1)-1$$

$$= 1$$

Since $f(1-0) = f(1+0) = f(1)$

So f is continuous at $x = 1$

As $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

So $$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$Lf'(1) = \lim_{h\to 0-0} \frac{1+h-1}{h}$$

$$= \lim_{h\to 0-0} \frac{h}{h}$$

$$= \lim_{h\to 0-0} (1)$$

$$Lf'(1) = 1$$

& $$Rf'(1) = \lim_{h\to 0+0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0+0} \frac{[2(1+h)-1]-1}{h}$$

$$= \lim_{h\to 0+0} \frac{2+2h-2}{h}$$

$$= \lim_{h\to 0+0} \frac{2h}{h}$$

$$= \lim_{h\to 0+0} (2)$$

$Rf'(1) = 2$ 8

Since $Lf'(1) \neq Rf'(1)$

So f is not differentiable at $x = 1$

Q3 If $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Show that f is Continuous & differentiable at $x=0$

Sol:- Given function is

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here $f(0) = 0$

Now $f(0-0) = \lim\limits_{x \to 0-0} f(x)$

$= \lim\limits_{x \to 0-0} x^2 \sin\frac{1}{x}$

Put $x = 0-h$
where $h>0$ & $h \to 0$

$= \lim\limits_{h \to 0} (-h)^2 \sin\left(\frac{1}{-h}\right)$

$= -\lim\limits_{h \to 0} h^2 \sin\frac{1}{h}$

$= -(0)^2 .$ Some no. in $[-1,1]$

$f(0-0) = 0$

& $f(0+0) = \lim\limits_{x \to 0+0} f(x)$

$= \lim\limits_{x \to 0+0} x^2 \sin\frac{1}{x}$

Put $x = 0+h$
where $h>0$ & $h \to 0$

$= \lim\limits_{h \to 0} h^2 \sin\frac{1}{h}$

$= (0)^2 .$ (Some no. in $[-1,1]$)

$f(0+0) = 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is continuous at $x = 0$

As $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$\Rightarrow f'(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$

Now $Lf'(0) = \lim\limits_{h\to 0-0} \dfrac{h^2 \sin\frac{1}{h} - 0}{h}$

$= \lim\limits_{h\to 0-0} \left(h \sin\frac{1}{h}\right)$

$= 0 \cdot$ Some no. in $[-1,1]$

$= 0$

& $Rf'(0) = \lim\limits_{h\to 0+0} \dfrac{f(h)-f(0)}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{h^2 \sin\frac{1}{h}}{h}$

$= \lim\limits_{h\to 0+0} \left(h \sin\frac{1}{h}\right)$

$= 0 \cdot$ Some no. in $[-1,1]$

$= 0$

Since $Lf'(0) = Rf'(0)$

So f is derivable at $x = 0$

<u>Q4</u> Is the function

$$f(x) = \begin{cases} (x-a)\cdot \sin\left(\frac{1}{x-a}\right) & x \neq a \\ 0 & x = a \end{cases}$$

Continuous & differentiable at $x = 0$

Sol. Given function is

$$f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & x \neq a \\ 0 & x = a \end{cases}$$

Here $f(a) = 0$

& $f(a-0) = \lim_{x \to a-0} f(x)$

$= \lim_{x \to a-0} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= \lim_{h \to 0} (a-h-a)\sin\left(\frac{1}{a-h-a}\right)$ Put $x = a-h$ where $h>0$ & $h \to 0$

$= \lim_{h \to 0} (-h)\sin\left(\frac{1}{-h}\right)$

$= \lim_{h \to 0}\left(h\sin\frac{1}{h}\right)$

$= 0 \cdot$ Some no. in $[-1,1]$

$f(a-0) = 0$

& $f(a+0) = \lim_{x \to a+0} f(x)$

$= \lim_{x \to a+0} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= \lim_{h \to 0} (a+h-a)\sin\left(\frac{1}{a+h-a}\right)$ Put $x = a+h$ where $h>0$ & $h \to 0$

$= \lim_{h \to 0} h\sin\frac{1}{h}$

$= 0 \cdot$ Some no. in $[-1,1]$

$f(a+0) = 0$

Since $f(a-0) = f(a+0) = f(a)$

So f is Continuous at $x = a$

As $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\Rightarrow \quad f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$$Lf'(a) = \lim_{h\to 0-0} \frac{(\cancel{a}+h-\cancel{a})\sin\left(\frac{1}{a+h-a}\right)-0}{h}$$

$$= \lim_{h\to 0-0} \frac{h\sin\frac{1}{h}}{h}$$

$$= \lim_{h\to 0-0} \sin\frac{1}{h}$$

$Lf'(a) =$ Some no. in $[-1,1]$

So $Lf'(a)$ does not exist.

Hence f is not derivable at $x=a$

Q5 Let $f(x) = \begin{cases} x\tan^{-1}\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

Discuss the Continuity & differentiability of f at $x=0$

Sol:- Given function is

$$f(x) = \begin{cases} x\tan^{-1}\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$$

Here $f(0) = 0$

& $f(0-0) = \lim_{x\to 0-0} f(x)$

$$= \lim_{x\to 0-0} x\tan^{-1}\frac{1}{x}$$

$=$ 0. Some real no.

So $f(0-0) = 0$

& $f(0+0) = \lim_{x \to 0+0} f(x)$

$= \lim_{x \to 0+0} x\tan^{-1}\frac{1}{x}$

$= 0 \cdot$ Some real no.

$= 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x = 0$

As $f'(x) = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

$\Rightarrow f'(0) = \lim_{h \to 0} \dfrac{f(h)-f(0)}{h}$

$Lf'(0) = \lim_{h \to 0-0} \dfrac{h\tan^{-1}\frac{1}{h} - 0}{h}$

$= \lim_{h \to 0-0} \tan^{-1}\frac{1}{h}$

which does not exist

So $Lf'(0)$ does not exist

Hence f is not derivable at $x = 0$

Q6 Examine for Continuity & differentiability the function $x^{4/3}$ at $x = 0$

Sol:- Given function is

$f(x) = x^{4/3}$

Here $f(0) = (0)^{4/3} = 0$

& $f(0-0) = \lim_{x \to 0-0} f(x)$

$= \lim_{x \to 0-0} x^{4/3}$

13

$f(0-0) = 0$

& $f(0+0) = \lim\limits_{x \to 0+0} x^{4/3}$

$= (0)^{4/3}$

$= 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x = 0$

As $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\Rightarrow f'(0) = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$Lf'(0) = \lim\limits_{h \to 0-0} \dfrac{h^{4/3} - 0}{h}$

$= \lim\limits_{h \to 0-0} h^{4/3 - 1}$

$= \lim\limits_{h \to 0-0} h^{1/3}$

$= 0$

& $Rf'(0) = \lim\limits_{h \to 0+0} \dfrac{f(h) - f(0)}{h}$

$= \lim\limits_{h \to 0+0} \dfrac{h^{4/3} - 0}{h}$

$= \lim\limits_{h \to 0+0} h^{1/3}$

$= (0)^{1/3}$

$= 0$

Since $Lf'(0) = Rf'(0)$

So f is derivable at $x = 0$

Q7 Find the values of a & b so that f is Continuous & differentiable at $x = 1$ where 14

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax+b & \text{if } x \geq 1 \end{cases}$$

Sol:- Given function is

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax+b & \text{if } x \geq 1 \end{cases}$$

Since f is Continuous at $x = 1$

So $\quad f(1-0) = f(1+0)$

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1+0} f(x)$$

$$\lim_{x\to 1-0} x^3 = \lim_{x\to 1+0} (ax+b)$$

$$(1)^3 = a(1)+b$$

$$\Rightarrow a+b = 1 \quad \text{———} \quad (1)$$

Also as f is derivable at $x = 1$

So $\quad Lf'(1) = Rf'(1)$

Now $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

So $\quad f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$$Lf'(1) = \lim_{h\to 0-0} \frac{(1+h)^3 - 1}{h}$$

$$= \lim_{h\to 0-0} \frac{\cancel{1}+3h+3h^2+h^3-\cancel{1}}{h}$$

$$= \lim_{h\to 0-0} (3+3h+h^2)$$

$$= 3+0+0$$

$$Lf'(1) = 3$$

$$\& \quad Rf'(1) = \lim_{h\to 0+0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0+0} \frac{[a(1+h)+b]-[a+b]}{h}$$

$$= \lim_{h\to 0+0} \frac{\cancel{a}+ah+\cancel{b}-\cancel{a}-\cancel{b}}{h}$$

$$= \lim_{h\to 0+0} \frac{ah}{h}$$

$$= \lim_{h\to 0+0} a$$

$$Rf'(1) = a$$

Since f is differentiable at $x = 1$

So $Lf'(1) = Rf'(1)$

$$\Rightarrow \quad 3 = a$$

or $\boxed{a = 3}$

Put in ①

$3 + b = 1$

$\boxed{b = 2}$

<u>Q8</u> Let $f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \le \pi/6 \\ ax+b & \text{if } \frac{\pi}{6} < x \le 1 \end{cases}$

Derive the values of a & b so that f is Continuous & differentiable at $x = \pi/6$

<u>Sol:-</u> Given function is

$$f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \le \pi/6 \\ ax+b & \text{if } \frac{\pi}{6} < x \le 1 \end{cases}$$

Since f is continuous at $x = \pi/6$

$$\Rightarrow f\left(\frac{\pi}{6} - 0\right) = f\left(\frac{\pi}{6} + 0\right)$$

or $\lim\limits_{x \to \frac{\pi}{6} - 0} f(x) = \lim\limits_{x \to \frac{\pi}{6} + 0} f(x)$

$$\lim_{x \to \frac{\pi}{6} - 0} \sin 2x = \lim_{x \to \frac{\pi}{6} + 0} (ax+b)$$

$$\sin 2\left(\frac{\pi}{6}\right) = a\left(\frac{\pi}{6}\right) + b$$

$$\sin \pi/3 = \frac{\pi a + 6b}{6}$$

$$\frac{\sqrt{3}}{2} = \frac{\pi a + 6b}{6}$$

$$2(\pi a + 6b) = 6\sqrt{3}$$

$$\pi a + 6b = 3\sqrt{3} \quad \text{—— (1)}$$

Also f is derivable at $x = \pi/6$

So $Lf'\left(\frac{\pi}{6}\right) = Rf'(\pi/6)$

Now $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

So $f'(\pi/6) = \lim\limits_{h \to 0} \dfrac{f\left(\frac{\pi}{6} + h\right) - f(\pi/6)}{h}$

$$Lf'(\pi/6) = \lim_{h \to 0-0} \frac{\sin 2\left(\frac{\pi}{6} + h\right) - \sin 2(\pi/6)}{h}$$

$$= \lim_{h \to 0-0} \frac{\sin\left(\frac{\pi}{3} + 2h\right) - \sin \pi/3}{h}$$

$$Lf'\left(\frac{\pi}{6}\right) = \lim_{h\to 0-0} \frac{2\cos\left(\frac{\pi/3+2h+\pi/3}{2}\right)\cdot\sin\left(\frac{\frac{\pi}{3}+2h-\pi/3}{2}\right)}{h}$$

$$= \lim_{h\to 0-0} 2\cos(\pi/3+h)\cdot\frac{\sin h}{h}$$

$$= \left[\lim_{h\to 0-0} 2\cos(\pi/3+h)\right]\left[\lim_{h\to 0-0}\frac{\sin h}{h}\right]$$

$$= 2\cos(\pi/3+0)\cdot 1$$

$$= 2\cos(\pi/3)$$

$$= 2\cdot\frac{1}{2}$$

$$Lf'(\pi/6) = 1$$

& $$Rf'(\pi/6) = \lim_{h\to 0+0}\frac{f(\pi/6+h)-f(\pi/6)}{h}$$

$$= \lim_{h\to 0+0}\frac{[a(\pi/6+h)+b]-[a(\pi/6)+b]}{h}$$

$$= \lim_{h\to 0+0}\frac{a\cancel{\pi/6}+ah\cancel{+b}-a\cancel{\pi/6}\cancel{-b}}{h}$$

$$= \lim_{h\to 0+0}\frac{a\cancel{h}}{\cancel{h}}$$

$$Rf'(\pi/6) = a$$

Since f is derivable at $x = \pi/6$

So $\quad Lf'(\pi/6) = Rf'(\pi/6)$

or $\quad 1 = a$

$\boxed{a=1}$ Put in (1)

$\pi(1)+6b = 3\sqrt{3}$

$\Rightarrow \quad 6b = 3\sqrt{3}-\pi$

$$\boxed{b = \frac{3\sqrt{3}-\pi}{6}}$$

Q9 If $f(x) = x\tanh\frac{1}{x}$, $x \neq 0$ & $f(0) = 0$. 18

Discuss the Continuity & differentiability of f at $x = 0$

Sol:- Given function is

$$f(x) = \begin{cases} x\tanh\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\text{or } f(x) = \begin{cases} x\cdot\dfrac{e^{\frac{1}{x}} - e^{-1/x}}{e^{1/x} + e^{-1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here $f(0) = 0$

& $$f(0-0) = \lim_{x\to 0-0} x\cdot\frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$$

$$= \lim_{x\to 0-0} x\left[\frac{e^{\frac{1}{x}} - \frac{1}{e^{1/x}}}{e^{1/x} + \frac{1}{e^{1/x}}}\right]$$

$$= \lim_{x\to 0-0} x\left[\frac{e^{\frac{2}{x}} - 1}{e^{2/x} + 1}\right]$$

$$= \lim_{h\to 0} (-h)\left[\frac{e^{-2/h} - 1}{e^{-2/h} + 1}\right]$$ Put $x = 0-h$ where $h > 0$ & $h \to 0$

$$= (0)\cdot\left[\frac{0-1}{0+1}\right]$$

$$= (0)(-1)$$

$$f(0-0) = 0$$

& $$f(0+0) = \lim_{x\to 0+0} f(x)$$

$$= \lim_{x\to 0+0} x\left[\frac{e^{\frac{2}{x}} - 1}{e^{2/x} + 1}\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{2/h}-1}{e^{2/h}+1}\right] \qquad \text{Put } x = 0+h \text{ where } h>0 \text{ \& } h\to 0$$ 19

$$= \lim_{h\to 0} h\left[\frac{1-e^{-2/h}}{1+e^{-2/h}}\right]$$

$$= (0)\left[\frac{1-0}{1+0}\right]$$

$f(0+0) = 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x = 0$

As $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

Now $$Lf'(0) = \lim_{h\to 0-0}\frac{h\left(\frac{e^{2/h}-1}{e^{2/h}+1}\right)}{h}$$

$$= \lim_{h\to 0-0}\left(\frac{e^{\frac{2}{h}}-1}{e^{\frac{2}{h}}+1}\right)$$

$$= \frac{0-1}{0+1}$$

$$Lf'(0) = -1$$

& $$Rf'(0) = \lim_{h\to 0+0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0+0}\frac{h\cdot\left(\frac{e^{\frac{2}{h}}-1}{e^{2/h}+1}\right)-0}{h}$$

$$= \lim_{h\to 0+0}\left(\frac{e^{\frac{2}{h}}-1}{e^{2/h}+1}\right)$$

$$Rf'(0) = \lim_{h \to 0+0} \left(\frac{1 - e^{-2/h}}{1 + e^{-2/h}} \right)$$

$$= \frac{1-0}{1+0}$$

$$Rf'(0) = 1$$

Since $Lf'(0) \neq Rf'(0)$

So f is not differentiable at $x = 0$

Find the slope of the tangent line to the given curve at the indicated pt. (Problems 10–12).

Q10 $y = x^2$ at $(2, 4)$

Sol:- Given eq. of curve is

$$y = x^2$$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2x$$

or $\frac{dy}{dx} = 2(2)$ at $(2,4)$

$$\Rightarrow \frac{dy}{dx} = 4$$

which is req. slope of tangent line to given curve

Q11 $y = \frac{1}{x}$ at $(1, 1)$

Sol:- Given eq. of curve is

$$y = \frac{1}{x}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

$\frac{dy}{dx} = \frac{-1}{(1)^2}$ at $(1,1)$

So $\frac{dy}{dx} = -1$, which is req. slope of tangent line to given curve.

Q12 $\quad y = x^2 - 7x + 3 \quad$ at $(7,3)$

Sol:- Given eq. of Curve is

$$y = x^2 - 7x + 3$$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2x - 7$$

or $\frac{dy}{dx} = 2(7) - 7 \quad$ at $(7,3)$

or $\frac{dy}{dx} = 14 - 7$

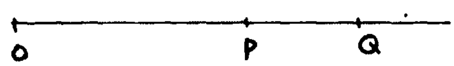
$$\frac{dy}{dx} = 7$$

which is the req. slope of tangent line to given Curve.

Q13 Let v be the velocity of a particle at any given time t. Deduce that the acceleration at this instant is $\frac{dv}{dt}$.

Sol:-

Suppose that a particle starts its motion along the line from fixed pt. O.



Let after time t, the particle reaches at the pt. P with vel. v at pt. P. Further suppose that after a small interval of time δt, it reaches at pt. Q with vel. δv. It means particle attains vel. δv in time δt in going from P to Q. Then avg acc. of particle $= \frac{\delta v}{\delta t}$

& acc. of particle $= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$ —— Ans.

Find the velocity & acc. at the end of 22.
0, 1, 2 seconds (Problems 14–16).

Q14 $S = \dfrac{1}{t+1}$

Sol:- Given

$$S = \frac{1}{t+1}$$

Diff. w.r.t. t

$$\frac{ds}{dt} = \frac{-1}{(t+1)^2}$$

or $v = \dfrac{-1}{(t+1)^2}$

Put $t = 0, 1, 2$

$$v_0 = \frac{-1}{(0+1)^2} = -1$$

$$v_1 = \frac{-1}{(1+1)^2} = -\frac{1}{4}$$

$$v_2 = \frac{-1}{(2+1)^2} = \frac{-1}{9}$$

As $v = \dfrac{-1}{(t+1)^2}$

$$v = -(t+1)^{-2}$$

Diff. w.r.t. t

$$\frac{dv}{dt} = 2(t+1)^{-3}$$

$$a = \frac{2}{(t+1)^3}$$

Put $t = 0, 1, 2$

$$a_0 = \frac{2}{(0+1)^3} = \frac{2}{1} = 2$$

$$a_1 = \frac{2}{(1+1)^3} = \frac{2}{(2)^3} = \frac{2}{8} = \frac{1}{4}$$

$$a_2 = \frac{2}{(2+1)^3} = \frac{2}{(3)^3} = \frac{2}{27}$$ —— Ans.

Q15 $S = t^2+2t+5$ 28

Sol.- Given

$S = t^2+2t+5$

Diff. w.r.t. t

$\frac{ds}{dt} = 2t+2$

$v = 2(t+1)$

Put $t = 0, 1, 2$

$v_0 = 2(0+1) = 2$

$v_1 = 2(1+1) = 4$

$v_2 = 2(2+1) = 6$

As $v = 2(t+1)$

Diff. w.r.t. t

$\frac{dv}{dt} = 2(1+0)$

$a = 2$

Put $t = 0, 1, 2$

$a_0 = 2$

$a_1 = 2$

$a_2 = 2$

Q16 $S = t^2(t-1)$

Sol.- Given

$S = t^2(t-1)$

$S = t^3-t^2$

Diff. w.r.t. t

$\frac{ds}{dt} = 3t^2-2t$

$v = 3t^2 - 2t$

Put $t = 0, 1, 2$

$v_0 = 3(0)^2 - 2(0) = 0$

$v_1 = 3(1)^2 - 2(1) = 3 - 2 = 1$

$v_2 = 3(2)^2 - 2(2) = 12 - 4 = 8$

As $v = 3t^2 - 2t$

Diff. w.r.t. t

$$\frac{dv}{dt} = 6t - 2$$

$a = 6t - 2$

Put $t = 0, 1, 2$

$a_0 = 6(0) - 2 = -2$

$a_1 = 6(1) - 2 = 4$

$a_2 = 6(2) - 2 = 10$

Q17 A pt. moves in a st. line so that its distance S (in meters) after time t (in seconds) is $S = 4t^2 - 16t + 12$

Find

(i) The average vel. in the interval $[1, 1+\Delta t]$

(ii) The velocity at $t = 1$

Sol:- Given

$$S = 4t^2 - 16t + 12$$

we know that average velocity of particle is

$$\frac{\Delta S}{\Delta t} = \frac{\text{total change in } S}{\text{total change in } t}$$

$$\frac{\Delta S}{\Delta t} = \frac{S(1+\Delta t) - S(1)}{1+\Delta t - 1}$$

$$= \frac{[4(1+\Delta t)^2 - 16(1+\Delta t) + 12] - [4(1)^2 - 16(1) + 12]}{\Delta t}$$

$$= \frac{4(1+2\Delta t+\Delta t^2) - 16 - 16\Delta t + 12 - 4 + 16 - 12}{\Delta t}$$

$$= \frac{\cancel{4} + 8\Delta t + 4\Delta t^2 - \cancel{16} - 16\Delta t + \cancel{12} - \cancel{4} + \cancel{16} - \cancel{12}}{\Delta t}$$

$$= \frac{4\Delta t^2 - 8\Delta t}{\Delta t}$$

$$= \frac{4\cancel{\Delta t}(\Delta t - 2)}{\cancel{\Delta t}}$$

$$= 4(\Delta t - 2)$$

$\frac{\Delta S}{\Delta t} = 4\Delta t - 8$ is the avg. vel. in interval $[1, 1+\Delta t]$

Now velocity of particle at $t = 1$ is

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0} (4\Delta t - 8) \qquad \text{at } t = 1$$

$$\frac{ds}{dt} = 4(0) - 8$$

So $v = -8\ m/sec$ at $t = 1$

Q18 The position of a body (in feet) at time t seconds is

$$S = t^3 - 5t^2 + 9t$$

Find the body's acc. each time its velocity is zero.

Sol:

Sol. Given that

$$S = t^3 - 6t^2 + 9t$$

Diff. w.r.t. t

$$\frac{ds}{dt} = 3t^2 - 12t + 9$$

$$v = 3t^2 - 12t + 9$$

Diff. w.r.t. t

$$\frac{dv}{dt} = 6t - 12$$

or $a = 6t - 12$

If vel. of body is zero

then $3t^2 - 12t + 9 = 0$

$$t^2 - 4t + 3 = 0$$

$$t^2 - 3t - t + 3 = 0$$

$$t(t-3) - 1(t-3) = 0$$

$$(t-3)(t-1) = 0$$

$\boxed{t = 1, 3}$ is the time at which vel. of particle is zero

Now acc. of body at $t = 1$ is

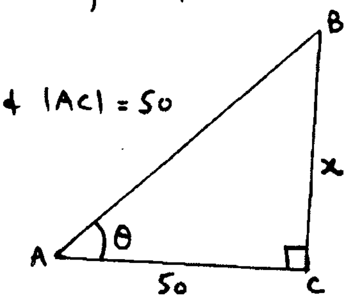
$$[a]_{t=1} = 6(1) - 12 = -6$$

& acc. of body at $t = 3$ is

$$[a]_{t=3} = 6(3) - 12 = 18 - 12 = 6$$

Q19 A ladder is placed 50 meters from a wall at an angle θ with the horisontal. The top of the ladder is x meters above the ground. If the bottom of the ladder is pushed towards the wall. Find the rate of change of x with respect to θ when $\theta = 45^\circ$.

Sol. Let AB be the ladder where $|BC| = x$ & $|AC| = 50$



Let $\angle CAB = \theta$

From right angled $\triangle ABC$

$$\tan\theta = \frac{x}{50}$$

$$\Rightarrow x = 50\tan\theta$$

Diff. w.r.t. θ

$$\frac{dx}{d\theta} = 50\sec^2\theta$$

$$\Rightarrow \left[\frac{dx}{d\theta}\right]_{\theta=\pi/4} = 50\sec^2\pi/4 = 50(\sqrt{2})^2 = 50\times 2$$

So $\left[\frac{dx}{d\theta}\right]_{\theta=\pi/4} = 100$ m/rad. is the rate of change of x w.r.t. θ

$$= \frac{100}{\frac{180}{\pi}}$$

$$= \frac{100\pi}{180}$$

$$= \frac{100\times 3.1416}{180}$$

$= 1.75$ m/degree is the rate of change of x w.r.t. θ at $\theta = 45^\circ$

Q20 The no. of litres of water in a tank t minutes after the water starts draining out of the tank is given by

$$f(t) = 200(30-t)^2$$

(i) What is the average rate at which the water flows out during the first 5 minutes?

(ii) How fast is the water running out at the end of 5 minutes?

Sol. Given that

$$f(t) = 200(30-t)^2$$

Then the average rate at which the water flows out during the first 5 minutes is

$$\frac{\Delta f}{\Delta t} = \frac{\text{total change in } f(t)}{\text{total change in } t}$$

$$= \frac{f(5)-f(1)}{5-1}$$

$$= \frac{200(30-5)^2 - 200(30-1)^2}{4}$$

$$= \frac{200(25)^2 - 200(29)^2}{4}$$

$$= \frac{200\left[(25)^2-(29)^2\right]}{4}$$

$$= 50\left[625-841\right]$$

$$= 50(-216)$$

$$= -10800$$

i.e., the average rate at which the water flows out during first 5 minutes is 10800 litres/minute.

Now as $f(t) = 200(30-t)^2$

Diff. w.r.t. t

$f'(t) = -400(30-t)$

Hence the rate at which the water runs out after 5 minutes is

$= f'(5)$

$= -400(30-5)$

$= -400 \times 25$

$= -10000$

$= 10000$ litres/minute

Q21 The height S (in feet) of a rocket t seconds after its launching is given by

$$S = -t^3 + 96t^2 + 195t + 10, \quad t \geq 0$$

(i) Find the velocity of rocket at any time t.

(ii) The velocity of the rocket when t = 0, 30, 50, 70 seconds. Interpret the results.

(iii) The max. height attained by the rocket.

Sol:- The height of a rocket t seconds after its launching is

$$S = -t^3 + 96t^2 + 195t + 10$$

Diff. w.r.t. t

$$\frac{ds}{dt} = -3t^2 + 192t + 195$$

or $v = -3t^2 + 192t + 195$ is the velocity of the rocket at any time t.

(ii) As the vel. v of rocket at any time t is

$v = -3t^2 + 192t + 195$ 30

When $t = 0$, $v = -3(0)^2 + 192(0) + 195 = 195$ ft/sec.

When $t = 30$, $v = -3(30)^2 + 192(30) + 195 = -2700 + 5760 + 195$

$= 3255$ ft/sec.

When $t = 50$, $v = -3(50)^2 + 192(50) + 195 = -7500 + 9600 + 195$

$= 2295$ ft/sec.

When $t = 70$, $v = -3(70)^2 + 192(70) + 195 = -14700 + 13440 + 195$

$= -1065$ ft/sec.

Now we will interpret the above results.

When the rocket is launched, its vel. is $= 195$ ft/s

At $t = 30$ sec. its velocity becomes 3255 ft/sec. It means the vel. of rocket increases after some time b/w $t = 0$ & $t = 30$

At $t = 50$ sec. its vel. becomes 2295 ft/sec. which means vel. of rocket after some time b/w $t = 30$ & $t = 50$ sec.

Now if the vel. of rocket

Q22 The rupee cost $C(x)$ of producing x washing machines is 318

$$C(x) = 2000 + 100x - 0.1x^2$$

(i) Find the marginal cost at $x = 100$.

(ii) Show that the marginal cost at $x = 100$ is approximately the cost of producing the 101st washing machine

Sol. Given cost function is

$$C(x) = 2000 + 100x - 0.1x^2$$

Diff. w.r.t. x

$$C'(x) = 100 - (0.1).2x$$

or $C'(x) = 100 - 0.2x$

So marginal cost at $x = 100$ is

$$C'(100) = 100 - 0.2(100)$$
$$= 100 - 20$$
$$= 80$$

So the marginal cost at $x = 100$ is 80Rs.

(ii) It is obvious that the cost of producing 101st washing machine is

$$C(101) - C(100)$$
$$= \left[2000 + 100(101) - 0.1(101)^2\right] - \left[2000 + 100(100) - 0.1(100)^2\right]$$
$$= (2000 + 10100 - 1020.10) - (2000 + 10000 - 1000)$$
$$= \cancel{2000} + 10100 - 1020.10 - \cancel{2000} - 10000 + 1000 = 80$$

Thus marginal cost at $x = 100$ is approximately the cost of producing 101st washing machine.

Q23 The revenue $R(x)$ (in rupees) of selling x units of desks is

$$R(x) = 2000\left(1 - \frac{1}{x+2}\right)$$

(i) Find the marginal revenue when x no. of desks are sold.

(ii) Use $R'(x)$ to estimate the increase that will result by selling the 9th desk.

Sol: The revenue $R(x)$ of selling x units of desks is

$$R(x) = 2000\left(1 - \frac{1}{x+2}\right)$$

Diff. w.r.t. x

$$R'(x) = 2000\left(0 + \frac{1}{(x+2)^2}\right)$$

$$\boxed{R'(x) = \frac{2000}{(x+2)^2}}$$

is the marginal revenue when x no. of desks are sold.

(ii) As $R'(x) = \dfrac{2000}{(x+2)^2}$

So the approximate increase in revenue that will result by selling the 9th desk is

$$= R'(8)$$

$$= \frac{2000}{(8+2)^2}$$

$$= \frac{2000}{100}$$

$$= 20 \text{ Rs.}$$

Q24 The cost $C(x)$ (in rupees) of producing x units of fans is 33

$$C(x) = 100x + 200000$$

& the revenue $R(x)$ (in rupees) of selling these x no. of fans is

$$R(x) = -0.02x^2 + 400x$$

Find the profit function $P(x)$ & the marginal profit at $x = 2000$. Calculate the actual profit realized from the sale of 2001st fan.

Sol. The cost function & the revenue function are given as

$$C(x) = 100x + 200000$$

& $$R(x) = -0.02x^2 + 400x$$

We know that the profit function will be the difference of $R(x)$ & $C(x)$.

i.e., $$P(x) = R(x) - C(x)$$

$$= (-0.02x^2 + 400x) - (100x + 200000)$$

$$= -0.02x^2 + 400x - 100x - 200000$$

$$P(x) = -0.02x^2 + 300x - 200000$$

So marginal profit is

$$P'(x) = -0.04x + 300$$

Marginal profit at $x = 2000$ is

$$P'(2000) = (-0.04)(2000) + 300$$

So $P'(2000) = -80+300$

$= 220$ Rs.

Now actual profit from the sale of 2001st fan

$= P(2001) - P(2000)$

$= \left[-0.02(2001)^2 + 300(2001) - 200000\right]$

$- \left[-0.02(2000)^2 + 300(2000) - 200000\right]$

$= (-0.02)\left[(2001)^2 - (2000)^2\right] + 300$

$= (-0.02)(4001) + 300$

$= -80.02 + 300$

$= 219.98$ Rs.

Hence the marginal profit at $x = 2000$ is nearly equal to the profit from the sale of 2001st fan.

EXERCISE 2.6 (NEW BOOK)

Function of several variables

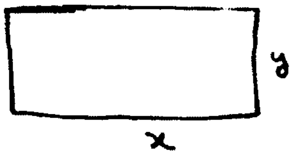
If $z = f(x, y)$

then z is called a fn. of two independent variables x & y.

Ex

Area of a rectangle

$$A = xy$$



Here A is a fn. of two variables x & y.

Similarly if

$$z = f(x, y, w)$$

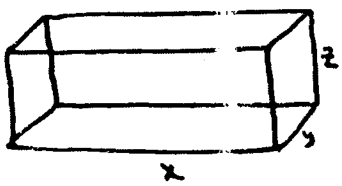
then z is called a fn. of three variables x, y & w

Ex

The volume of a rectangular parallelepipped with dimensions x, y & z is

$$V = xyz$$

Here V is a fn. of three variables x, y & z.



Similarly we can define fn. of several variables

Limit of a function

A function $z = f(x,y)$ is said to tend to a limit l as $(x,y) \to (a,b)$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x,y) - l| < \epsilon \quad \text{whenever} \quad |x-a| < \delta, \ |y-b| < \delta$$

for all pts. (x,y) other than (a,b)

We write it as

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

Note

If in $\lim\limits_{(x,y)\to(a,b)} f(x,y)$

we get two or more different values as $(x,y) \to (a,b)$ along different paths then

$\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

This path may be a line or plane curve through the pt. (a,b).

Partial derivatives

let $z = f(x,y)$

$$z+\delta z = f(x+\delta x, y)$$

$$\delta z = f(x+\delta x, y) - f(x,y)$$

Dividing both sides by δx

$$\frac{\delta z}{\delta x} = \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$\lim_{\delta x\to 0} \frac{\delta z}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x,y)}{\delta x}$$

If the limit on R.H.S. exists as a finite & definite quantity then it is called partial derivative of z (or of f) w.r.t. x & is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

So $$\frac{\partial z}{\partial x} = \lim_{\delta x\to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

Similarly
$$\frac{\partial z}{\partial y} = \lim_{\delta y\to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

By def.
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

& $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

Partial derivatives of higher orders:

Let $z = f(x,y)$ then

higher derivatives of $f(x,y)$ are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

Implicit function

A function f of two variables of the form

$$f(x,y) = 0$$

is called an implicit fn.

Now in this function

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

or $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

EXERCISE 2.6 (NEW BOOK)

Exercise 2.5

Evaluate the given limit (Problems 1–5):

Q1 $\lim\limits_{(x,y)\to(0,0)} \dfrac{5-x^2}{4+x+y}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{5-x^2}{4+x+y}$$

$$= \frac{\lim\limits_{(x,y)\to(0,0)} (5-x^2)}{\lim\limits_{(x,y)\to(0,0)} (4+x+y)}$$

$$= \frac{5-0}{4+0+0}$$

$$l = \frac{5}{4}$$

Q2 $\lim\limits_{(x,y)\to(1,-1)} e^{-xy}$

Sol.

Let l be the req. limit then

$$l = \lim_{(x,y)\to(1,-1)} e^{-xy}$$

$$= e^{(-1)(-1)}$$

$$= e^{1}$$

$$l = e$$

Q3 $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}\sin xy}{xy}$

Sol:- Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy}$$

$$= \lim_{(x,y)\to(0,0)} \left[e^{xy} \cdot \frac{\sin xy}{xy}\right]$$

$$= \left[\lim_{(x,y)\to(0,0)} e^{xy}\right]\left[\lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy}\right]$$

$$= e^{0} \cdot 1$$

$$= 1 \cdot 1$$

$$l = 1$$

Q4 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$

Sol:- Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

Put $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right]$

$\Rightarrow x^2+y^2 = r^2$ As $(x,y)\to(0,0)$, $r\to 0$

So above limit becomes

$$l = \lim_{r\to 0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2}$$

$$l = \lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

$$= \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta)$$

$$= 0(\cos^3\theta - \sin^3\theta)$$

$$l = 0$$

Q5 $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^3 - 2xy + 3x^2 - 2y}{x^2y + 4y^2 - 6x^2 + 24y}$

Sol. let l be the req. limit then

$$l = \lim_{(x,y)\to(2,2)} \frac{x^3 - 2xy + 3x^2 - 2y}{x^2y + 4y^2 - 6x^2 + 24y}$$

$$= \frac{\lim_{(x,y)\to(2,2)} (x^3 - 2xy + 3x^2 - 2y)}{\lim_{(x,y)\to(2,2)} (x^2y + 4y^2 - 6x^2 + 24y)}$$

$$= \frac{8 - 8 + 12 - 4}{8 + 16 - 24 + 48}$$

$$= \frac{8}{48}$$

$$l = \frac{1}{6}$$

In problems 6–10, show that the given limit does not exist

Q6 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Sol. Let l be the req. limit then 75

$$l = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

If we show that $f(x,y) = \dfrac{xy}{x^2+y^2}$ approaches to different values as $(x,y) \to (0,0)$ from different directions then we say that limit does not exist.

Let $(x,y) \longrightarrow (0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{x(mx)}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{m}{1+m^2}\right)$$

which is different for different values of m

So the given limit does not exist.

Q7 $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

If we show that $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ approaches to different values as $(x,y) \to (0,0)$ from different directions then we say that limit does not exist.

let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(1-m^2)}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{1-m^2}{1+m^2}\right)$$

which is different for different values of m

So the given limit does not exist.

Q8 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{ax^2+by}{cy^2+dx}$

Sol:- let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{ax^2+by}{cy^2+dx}$$

If we show that $f(x,y) = \dfrac{ax^2+by}{cy^2+dx}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \left(\frac{ax^2+bmx}{cm^2x^2+dx}\right)$$

$$l = \lim_{(x,y)\to(0,0)} \left(\frac{ax+bm}{cm^2x+d}\right)$$

$$= \frac{0+bm}{0+d}$$

$$l = \frac{bm}{d}$$

which is different for different values of m

So the given limit does not exist.

Q9 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^4+y^4}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^4+y^4}$$

If we show that $f(x,y) = \dfrac{(x^2+y^2)^2}{x^4+y^4}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{(x^2+m^2x^2)^2}{x^4+m^4x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4(1+m^2)^2}{x^4(1+m^4)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(1+m^2)^2}{1+m^4}$$

viz different for different values of m 78

So the given limit does not exist.

Q10 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Sol. let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

If we show that $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{x\,m^2x^2}{x^2+m^4x^4}$$

$$l = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^4x^2}$$

So $l \to 0$ as $(x,y)\to(0,0)$

Hence along every st. line through origin

$f(x,y) \longrightarrow (0,0)$

Next

Suppose $(x,y)\to(0,0)$ along $x = y^2$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{y^2.y^4}{y^4+y^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y^4}{2y^4}$$

$$l = \lim_{(x,y)\to(0,0)} \frac{1}{2}$$ 79

So $l \to \frac{1}{2}$ as $(x,y)\to(0,0)$ along parabola $x = y^2$

Hence the limit does not exist.

Q11 Let $f(x,y) = \dfrac{xy^2}{x^3+y^3}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Show that f is not Continuous at the origin.

Sol Given function is

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^3+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 0$ ——(given)

Now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3}$$

let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x\,m^2x^2}{x^3+m^3x^3}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m^2}{1+m^3}$$

viz different for different values of m

So $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

Hence given function is not Continuous at $(0,0)$

Q12 Find a such that the function

$$f(x,y) = \frac{3xy}{\sqrt{x^2+y^2}} \quad \text{if } (x,y) \neq (0,0)$$
$$= a \quad \text{if } (x,y) = (0,0)$$

is Continuous at $(0,0)$.

Sol:- Given function is

$$f(x,y) = \begin{cases} \dfrac{3xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = a$ (given)

Now $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}}$$

let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x(mx)}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{3mx}{\sqrt{1+m^2}}$$

So $$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Since $f(x,y)$ is Continuous at $(0,0)$

So $$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$$\Rightarrow \quad 0 = a$$

or $\underline{a = 0}$

Q13 Let $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Examine the Continuity of f at $(0,0)$.

Do $f_x(0,0)$ & $f_y(0,0)$ exist.

Sol. Given function is

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 0$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

Let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+m^3x^3}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x+m^3x}{1+m^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(1+m^3)x}{1+m^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Since $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

So $f(x,y)$ is Continuous at $(0,0)$

Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= \lim_{h\to 0} (1)$$

So $f_x(0,0) = 1$

&

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,K)-f(0,0)}{K}$$

$$= \lim_{K\to 0} \frac{K-0}{K}$$

$$= \lim_{K\to 0} \frac{K}{K}$$

$$= \lim_{K\to 0} 1$$

$f_y(0,0) = 1$

So both $f_x(0,0)$ & $f_y(0,0)$ exist

Q14 let $f(x,y) = \dfrac{x^2 y}{x^4+y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Prove that f is not Continuous at $(0,0)$.

Do $f_x(0,0)$ & $f_y(0,0)$ exist

Sol. Given fn. is

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 0$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$$

Let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^4+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

But if $(x,y) \to (0,0)$ along the Curve $x^2 = y$

then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(x^2)}{x^4+x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$$

So the unique limit does not exist.

Hence f is not continuous at $(0,0)$

Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

& $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k}$$

$$= \lim_{k\to 0} 0$$

$$= 0$$

So both $f_x(0,0)$ & $f_y(0,0)$ exist.

Find the first order partial derivatives of the given function (Problems 15 – 22).

Q15 $\quad f(x,y) = x^{y^2}$

Sol. Given

$$f(x,y) = x^{y^2}$$

Diff. partially w.r.t. x & y

$$f_x = y^2 \cdot x^{y^2-1}$$

$$f_y = x^{y^2} \cdot \ln x \cdot 2y$$

Q16 $f(x,y) = e^{x^2+y^2}$

Sol:- Given

$$f(x,y) = e^{x^2+y^2}$$

Diff. partially w.r.t. x & y

$$f_x = e^{x^2+y^2} \cdot 2x$$

$$= 2x\, e^{x^2+y^2}$$

$$f_y = e^{x^2+y^2} \cdot 2y$$

$$= 2y \cdot e^{x^2+y^2}$$

Q17 $f(x,y) = \tan^{-1}(y/x)$

Sol.

Given $f(x,y) = \tan^{-1}(y/x)$

Diff. partially w.r.t. x & y

$$f_x = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}(y/x)$$

$$= \frac{1}{1+\frac{y^2}{x^2}} \cdot -\frac{y}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2}$$

$$f_x = \frac{-y}{x^2+y^2}$$

$$\& \quad f_y = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial y}(y/x)$$

$$= \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$f_y = \frac{x}{x^2+y^2}$$

Q18 $f(x,y) = \tan^{-1}(x+y)$

Sol. Given

$$f(x,y) = \tan^{-1}(x+y)$$

Diff. partially w.r.t. x & y

$$f_x = \frac{1}{1+(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y)$$

$$= \frac{1}{1+(x+y)^2} \cdot 1$$

$$f_x = \frac{1}{1+(x+y)^2}$$

& $f_y = \dfrac{1}{1+(x+y)^2} \cdot \dfrac{\partial}{\partial y}(x+y)$

$= \dfrac{1}{1+(x+y)^2} \cdot 1$

$f_y = \dfrac{1}{1+(x+y)^2}$

Q19 $f(x,y) = e^{ax}\sin by$

Sol.- Given

$f(x,y) = e^{ax}\sin by$

Diff. partially w.r.t. x & y

$f_x = (e^{ax} \cdot a)(\sin by)$

$= a e^{ax} \cdot \sin by$

& $f_y = e^{ax} \cdot \cos by \cdot b$

$= b e^{ax} \cos by$

Q20 $f(x,y) = \ln(x^2+y^2)$

Sol.- Given

$f(x,y) = \ln(x^2+y^2)$

Diff. partially w.r.t. x & y

$f_x = \dfrac{1}{x^2+y^2} \cdot \dfrac{\partial}{\partial x}(x^2+y^2)$

$= \dfrac{1}{x^2+y^2}(2x)$

88

$$f_x = \frac{2x}{x^2+y^2}$$

& $$f_y = \frac{1}{(x^2+y^2)} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot 2y$$

$$= \frac{2y}{x^2+y^2}$$

Q21 $$f(x,y) = \ln\left[\sqrt{\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}}\right]$$

Sol:- Given

$$f(x,y) = \ln\left[\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$$

$$f(x,y) = \ln\left(\sqrt{x^2+y^2}-x\right) - \ln\left(\sqrt{x^2+y^2}+x\right)$$

Diff. partially w.r.t. x & y

$$f_x = \frac{1}{(\sqrt{x^2+y^2}-x)} \cdot \left(\frac{1}{2\sqrt{x^2+y^2}} \cdot 2x - 1\right) - \frac{1}{(\sqrt{x^2+y^2}+x)} \cdot \left(\frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + 1\right)$$

$$= \frac{1}{(\sqrt{x^2+y^2}-x)}\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right) - \frac{1}{(\sqrt{x^2+y^2}+x)}\left(\frac{x}{\sqrt{x^2+y^2}} + 1\right)$$

$$= \frac{(x-\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2})} - \frac{(x+\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2}+x)(\sqrt{x^2+y^2})}$$

$$f_x = \frac{-1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}} = -\frac{2}{\sqrt{x^2+y^2}}$$

$$+ f_y = \frac{1}{(\sqrt{x^2+y^2}-x)} \cdot \left[\frac{1}{2\sqrt{x^2+y^2}} \cdot 2y\right] - \frac{1}{(\sqrt{x^2+y^2}+x)} \cdot \left[\frac{1}{2\sqrt{x^2+y^2}} \cdot 2y\right]$$

$$= \frac{y}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2})} - \frac{y}{(\sqrt{x^2+y^2}+x)(\sqrt{x^2+y^2})}$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{1}{\sqrt{x^2+y^2}-x} - \frac{1}{\sqrt{x^2+y^2}+x}\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{\cancel{\sqrt{x^2+y^2}} + x - \cancel{\sqrt{x^2+y^2}} + x}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)}\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{2x}{\cancel{x^2}+y^2-\cancel{x^2}}\right]$$

$$= \frac{2xy}{\sqrt{x^2+y^2} \cdot y^2}$$

$$= \frac{2x}{y\sqrt{x^2+y^2}}$$

<u>Q22</u> $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

<u>Sol.</u> Given

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$$

Diff. partially w.r.t. x, y & z

$$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= \frac{-1}{2(x^2+y^2+z^2)^{3/2}} \cdot 2x$$

90

$$f_x = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

Now

$$f_y = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot \frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{-1}{2(x^2+y^2+z^2)^{3/2}} \cdot 2y$$

$$f_y = -\frac{y}{(x^2+y^2+z^2)^{3/2}}$$

& $$f_z = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot \frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2z$$

$$f_z = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Find the second order partial derivatives (Problems 23 – 26):

<u>Q23</u> e^{x-y}

<u>Sol.</u> let $z = e^{x-y}$ ———— (1)

Diff. (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = e^{x-y} \cdot 1$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x-y} \cdot 1$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = e^{x-y}}$$

Now $$\frac{\partial z}{\partial y} = e^{x-y} \cdot (-1)$$

$$\partial z/\partial y = -e^{x-y}$$

$$\frac{\partial^2 z}{\partial y^2} = -e^{x-y}\cdot(-1)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = e^{x-y}}$$

Now

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-e^{x-y}\right)$$

$$= -e^{x-y}\cdot 1$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = -e^{x-y}}$$

Q24 $\quad \dfrac{x+y}{x-y}$

Sol.

let $z = \dfrac{x+y}{x-y}$ ——— ①

Diff. ① partially w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{(x-y).1-(x+y).1}{(x-y)^2}$$

$$= \frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2}$$

$$\frac{\partial z}{\partial x} = -\frac{2y}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y\cdot\frac{-2}{(x-y)^3}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{4y}{(x-y)^3}}$$

Now Diff. ① partially w.r.t. y

92

$$\frac{\partial z}{\partial y} = \frac{(x-y).1-(x+y).(-1)}{(x-y)^2}$$

$$= \frac{x-\cancel{y}+x+\cancel{y}}{(x-y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2x.\frac{-2}{(x-y)^3}(-1)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = \frac{4x}{(x-y)^3}}$$

Now

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2x}{(x-y)^2}\right)$$

$$= 2\left[\frac{(x-y)^2.1-x.2(x-y)}{(x-y)^4}\right]$$

$$= 2\left(\frac{x^2-2\cancel{xy}+y^2-2x^2+2\cancel{xy}}{(x-y)^4}\right)$$

$$= 2\left[\frac{y^2-x^2}{(x-y)^4}\right]$$

$$= \frac{-2(x^2-y^2)}{(x-y)^4}$$

$$= \frac{-2(x-y)(x+y)}{(x-y)^4}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{-2(x+y)}{(x-y)^3}}$$

Q25 e^{x^y}

Sol.

Let $z = e^{x^y}$ ——— (1)

Diff. (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = e^{x^y} \cdot y x^{y-1}$$

$$\frac{\partial z}{\partial x} = y x^{y-1} \cdot e^{x^y}$$

$$\& \quad \frac{\partial^2 z}{\partial x^2} = y\left[x^{y-1} \cdot e^{x^y} \cdot y x^{y-1} + e^{x^y} \cdot (y-1) x^{y-2} \right]$$

$$= y\left[y x^{2y-2} \cdot e^{x^y} + (y-1) \cdot x^{y-2} \cdot e^{x^y} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x^y}\left[y^2 \cdot x^{2y-2} + y(y-1) \cdot x^{y-2} \right]$$

Now Diff. (1) w.r.t. y

$$\frac{\partial z}{\partial y} = e^{x^y} \cdot x^y \ln x$$

$$\frac{\partial^2 z}{\partial y^2} = \ln x \cdot \frac{\partial}{\partial y}\left(e^{x^y} \cdot x^y \right)$$

$$= \ln x\left[e^{x^y} \cdot x^y \cdot \ln x + x^y \cdot e^{x^y} \cdot x^y \ln x \right]$$

$$\frac{\partial^2 z}{\partial y^2} = e^{x^y}\left[x^y (\ln x)^2 + x^{2y} \cdot (\ln x)^2 \right]$$

Now

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y} \right)$$

$$= \frac{\partial}{\partial x}\left(e^{x^y} \cdot x^y \ln x \right)$$

$$= e^{x^y} \cdot x^y \cdot \frac{1}{x} + e^{x^y} \cdot \ln x \cdot y x^{y-1} + x^y \cdot \ln x \cdot e^{x^y} \cdot y x^{y-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{x^y}\left[x^{y-1} + y x^{y-1} \cdot \ln x + y x^{2y-1} \cdot \ln x \right]$$

Q26 $\tan(\tan^{-1}x+\tan^{-1}y)$

Sol.

Let $z=\tan(\tan^{-1}x+\tan^{-1}y)$

or $z=\dfrac{\tan(\tan^{-1}x)+\tan(\tan^{-1}y)}{1-\tan(\tan^{-1}x)\cdot\tan(\tan^{-1}y)}$

$$z=\frac{x+y}{1-xy} \quad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial z}{\partial x}=\frac{(1-xy)(1)-(x+y).(-y)}{(1-xy)^2}$$

$$=\frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\frac{\partial z}{\partial x}=\frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial^2 z}{\partial x^2}=(1+y^2).\frac{-2}{(1-xy)^3}.(-y)$$

$$\boxed{\frac{\partial^2 z}{\partial x^2}=\frac{2y(1+y^2)}{(1-xy)^3}}$$

Diff. (1) partially w.r.t. y

$$\frac{\partial z}{\partial y}=\frac{(1-xy).1-(x+y)(-x)}{(1-xy)^2}$$

$$=\frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$\frac{\partial z}{\partial y}=\frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial^2 z}{\partial y^2}=(1+x^2).\frac{-2}{(1-xy)^3}.(-x)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = \frac{2x(1+x^2)}{(1-xy)^3}}$$

Now

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1+x^2}{(1-xy)^2}\right)$$

$$= \frac{(1-xy)^2 \cdot 2x - (1+x^2)\cdot 2(1-xy)(-y)}{(1-xy)^4}$$

$$= \frac{2x(1-xy) + 2y(1+x^2)}{(1-xy)^3}$$

$$= \frac{2x - \cancel{2x^2y} + 2y + \cancel{2x^2y}}{(1-xy)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{2(x+y)}{(1-xy)^3}$$

In problems 27 – 32, verify that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

Q27 $f(x,y) = e^{xy}\cos(bx+c)$ ——— ①

Diff. ① partially w.r.t. x

$$\frac{\partial f}{\partial x} = e^{xy}\cdot -\sin(bx+c)\cdot b + \cos(bx+c)\cdot e^{xy}\cdot y$$

$$\frac{\partial f}{\partial x} = e^{xy}\left[y\cos(bx+c) - b\sin(bx+c)\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\cdot e^{xy}\left[y\cos(bx+c) - b\sin(bx+c)\right]$$

$$= e^{xy}\left[\cos(bx+c)\right] + \left[y\cos(bx+c) - b\sin(bx+c)\right] \cdot e^{xy} \cdot x$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{xy}\left[\cos(bx+c) + xy\cos(bx+c) - bx\sin(bx+c)\right] \text{ ——— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = x e^{xy} \cdot \cos(bx+c)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x e^{xy} \cos(bx+c)\right]$$

$$= x e^{xy} \cdot -\sin(bx+c) \cdot b + x\cos(bx+c) \cdot e^{xy} \cdot y + e^{xy} \cdot \cos(bx+c) \cdot 1$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy}\left[\cos(bx+c) + xy\cos(bx+c) - b\sin(bx+c)\right] \text{ ——— (B)}$$

From (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q28 $\quad f(x,y) = \ln(e^x + e^y)$

Sol.- Given

$$f(x,y) = \ln(e^x + e^y) \text{ ——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{(e^x + e^y)} \cdot e^x$$

$$\frac{\partial f}{\partial x} = \frac{e^x}{(e^x + e^y)}$$

Now

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{e^x}{(e^x+e^y)}\right)$$

$$= e^x \cdot \frac{-1}{(e^x+e^y)^2} \cdot e^y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{e^{x+y}}{(e^x+e^y)^2} \quad \text{———} \quad (A)$$

Now diff. (1) w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{(e^x+e^y)} \cdot e^y$$

$$\frac{\partial f}{\partial y} = \frac{e^y}{(e^x+e^y)}$$

Now

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{e^y}{(e^x+e^y)}\right)$$

$$= e^y \cdot \frac{-1}{(e^x+e^y)^2} \cdot e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{e^{x+y}}{(e^x+e^y)^2} \quad \text{———} \quad (B)$$

from (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q29 $f(x,y) = \ln\left(\frac{x^2+y^2}{xy}\right)$ 98

Sol. Given

$$f(x,y) = \ln\left(\frac{x^2+y^2}{xy}\right)$$

$$f(x,y) = \ln(x^2+y^2) - \ln(xy) \quad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}\cdot 2x - \frac{1}{xy}\cdot y$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{2x}{x^2+y^2} - \frac{1}{x}\right]$$

$$= 2x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{4xy}{(x^2+y^2)^2} \quad \text{——— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{1}{xy}\cdot x$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{2y}{x^2+y^2} - \frac{1}{y}\right]$$

$$= 2y \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4xy}{(x^2+y^2)^2} \text{ ——— } (B)$$

from (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q30 $f(x,y) = x^y + y^x$

Sol.- Given

$$f(x,y) = x^y + y^x \text{ ——— } (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = y\,x^{y-1} + y^x \cdot \ln y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[y\,x^{y-1} + y^x \ln y\right]$$

$$= y\cdot(x^{y-1}\cdot \ln x) + x^{y-1}\cdot 1 + y^x \cdot \frac{1}{y} + \ln y \cdot x\,y^{x-1}$$

$$\frac{\partial^2 f}{\partial y \partial x} = x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y) \text{ ——— } (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = x^y \ln x + x\,y^{x-1}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x^y \ln x + x\,y^{x-1}\right]$$

$$= x^y \cdot \frac{1}{x} + \ln x \cdot y\,x^{y-1} + x \cdot y^{x-1} \cdot \ln y + y^{x-1} \cdot 1$$

$$= x^{y-1} + \ln x \cdot y x^{y-1} + x y^{x-1} \ln y + y^{x-1}$$ 100

$$= x^{y-1}(1 + y\ln x) + y^{x-1}(x\ln y + 1)$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y) \quad \text{———} \quad (B)$$

from (A) + (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q31 $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$

Sol.- Given

$$f(x,y) = \sin^{-1}(x/y) \quad \text{———} \quad (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y}$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{y}} \cdot \frac{1}{y}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y} \frac{1}{\sqrt{y^2-x^2}}$$

$$= -\frac{1}{2}(y^2-x^2)^{-3/2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = -\frac{y}{(y^2-x^2)^{3/2}} \quad \text{———} \quad (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot -\frac{x}{y^2}$$

$$= \frac{1}{\sqrt{1-x^2/y^2}} \cdot \frac{-x}{y^2}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{-x}{y^2}$$

$$\frac{\partial f}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{-x}{y\sqrt{y^2-x^2}}\right]$$

$$= -\frac{1}{y}\cdot\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{y^2-x^2}}\right)$$

$$= -\frac{1}{y}\left[\frac{\sqrt{y^2-x^2}\cdot 1 - x\cdot\frac{(-2x)}{2\sqrt{y^2-x^2}}}{y^2-x^2}\right]$$

$$= -\frac{1}{y}\left[\frac{(y^2-x^2)+x^2}{(y^2-x^2)\sqrt{y^2-x^2}}\right]$$

$$= -\frac{1}{y}\left[\frac{y^2}{(y^2-x^2)^{3/2}}\right]$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -\frac{y}{(y^2-x^2)^{3/2}} \quad \text{———} \quad (B)$$

From (A) & (B)

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$$

Q32 $f(x,y) = \dfrac{xy}{\sqrt{1+x^2+y^2}}$

Sol.- Given

$$f(x,y) = \frac{xy}{\sqrt{1+x^2+y^2}} \qquad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = y\cdot\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2}}\right]$$

$$= y\left[\frac{\sqrt{1+x^2+y^2}\cdot 1 - x\cdot\frac{1}{2\sqrt{1+x^2+y^2}}\cdot 2x}{(1+x^2+y^2)}\right]$$

$$= y\left[\frac{\sqrt{1+x^2+y^2} - \frac{x^2}{\sqrt{1+x^2+y^2}}}{(1+x^2+y^2)}\right]$$

$$= y\left[\frac{1+\cancel{x^2}+y^2-\cancel{x^2}}{(1+x^2+y^2)^{3/2}}\right]$$

$$\frac{\partial f}{\partial x} = \frac{y+y^3}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{(y+y^3)}{(1+x^2+y^2)^{3/2}}\right]$$

$$= \frac{(1+x^2+y^2)^{3/2}\cdot(1+3y^2) - (y+y^3)\cdot\frac{3}{\cancel{2}}(1+x^2+y^2)^{1/2}\cdot\cancel{2}y}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)\cdot(1+3y^2) - 3(y^2+y^4)}{(1+x^2+y^2)}$$

$$= \frac{1+x^2+y^2+\cancel{3y^2}+3x^2y^2+\cancel{3y^4}-\cancel{3y^2}-\cancel{3y^4}}{(1+x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^3} \quad \text{———} \; (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = x \cdot \frac{\partial}{\partial y}\left[\frac{y}{\sqrt{1+x^2+y^2}}\right]$$

$$= x \cdot \left[\frac{\sqrt{1+x^2+y^2}\cdot 1 - y \cdot \frac{1}{\cancel{2}\sqrt{1+x^2+y^2}} \cdot \cancel{2}y}{(1+x^2+y^2)}\right]$$

$$= x\left[\frac{\sqrt{1+x^2+y^2} - \frac{y^2}{\sqrt{1+x^2+y^2}}}{(1+x^2+y^2)}\right]$$

$$= x\left[\frac{1+x^2+\cancel{y^2}-\cancel{y^2}}{(1+x^2+y^2)^{3/2}}\right]$$

$$\frac{\partial f}{\partial y} = \frac{x+x^3}{(1+x^2+y^2)^{3/2}}$$

Now

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{x+x^3}{(1+x^2+y^2)^{3/2}}\right]$$

$$= \frac{(1+x^2+y^2)^{3/2}\cdot(1+3x^2) - (x+x^3)\cdot\frac{3}{\cancel{2}}(1+x^2+y^2)^{1/2}\cdot \cancel{2}x}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)^{3/2}\cdot(1+3x^2) - 3x(x+x^3)(1+x^2+y^2)^{1/2}}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)(1+3x^2) - 3x(x+x^3)}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1+x^2+y^2+\cancel{3x^2}+\cancel{3x^4}+3x^2y^2-\cancel{3x^2}-\cancel{3x^4}}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^{3/2}} \quad \text{——— (B)}$$

From (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Show that each of the following functions satisfies Laplace's eq. $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (Problems 33-36)

<u>Q33</u> $f(x,y) = \sin x . \sinh y$

<u>Sol</u> Given

$$f(x,y) = \sin x . \sinh y \quad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \cos x . \sinh y$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x . \sinh y \quad \text{——— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \sin x \cosh y$$

$$\frac{\partial^2 f}{\partial y^2} = \sin x \sinh y \quad \text{——— (B)}$$

Adding (A) & (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

105

Q34 $f(x,y) = e^{-x}\cos y$

Sol. Given

$$f(x,y) = e^{-x}\cos y \quad \text{———} \quad (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = -e^{-x}\cos y$$

$$\frac{\partial^2 f}{\partial x^2} = -e^{-x}(-1).\cos y$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-x}\cos y \quad \text{———} \quad (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = -e^{-x}.\sin y$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y \quad \text{———} \quad (B)$$

Adding (A) & (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Q35 $f(x,y) = \ln\sqrt{x^2+y^2}$

Sol. Given

$$f(x,y) = \ln\sqrt{x^2+y^2}$$

$$f(x,y) = \ln(x^2+y^2)^{1/2}$$

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2) \quad \text{———} \quad (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}.2x$$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{------ (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{2} \cdot \frac{1}{(x^2+y^2)} \cdot 2y$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$

Now

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{------ (B)}$$

Adding (A) + (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Q36 $f(x,y) = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

Sol. Given

$$f(x,y) = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right) \quad \text{------ (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \cdot 2y\left[\frac{(x^2-y^2).1 - x(2x)}{(x^2-y^2)^2}\right]$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2} \cdot 2y\left[\frac{x^2-y^2-2x^2}{\cancel{(x^2-y^2)^2}}\right]$$

$$= \frac{2y(-x^2-y^2)}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{-2y(x^2+y^2)}{x^4+y^4+2x^2y^2}$$

$$= \frac{-2y(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x} = \frac{-2y}{(x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial x^2} = (-2y) \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2} \quad \text{———} \quad (A)$$

Now diff. ① partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \cdot 2x\left[\frac{(x^2-y^2).1 - y(-2y)}{(x^2-y^2)^2}\right]$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2+4x^2y^2} \cdot 2x\left[\frac{x^2-y^2+2y^2}{(x^2-y^2)^2}\right]$$

$$= \frac{2x(x^2+y^2)}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{2x(x^2+y^2)}{x^4+y^4+2x^2y^2}$$

$$= \frac{2x(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{4xy}{(x^2+y^2)^2} \qquad \text{------} (B)$$

Adding (A) + (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Q37 If $f(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$ then show that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

Sol:- Given

$$f(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$$

Diff. partially w.r.t. y

$$\frac{\partial f}{\partial y} = x^2 \cdot \frac{1}{1+y^2/x^2} \cdot \frac{1}{x} - \left[y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2} + \tan^{-1}(x/y) \cdot 2y\right]$$

$$= x^2 \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} - \left[\frac{-xy^2}{y^2+x^2} + 2y\tan^{-1}(x/y) \right]$$ 1.9

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}(x/y)$$

$$= \frac{x^3+xy^2}{x^2+y^2} - 2y\tan^{-1}(x/y)$$

$$= \frac{x(x^2+y^2)}{(x^2+y^2)} - 2y\tan^{-1}(x/y)$$

$$\frac{\partial f}{\partial y} = x - 2y\tan^{-1}(x/y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}(x/y) \right]$$

$$= 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2}{1+\frac{x^2}{y^2}}$$

$$= 1 - \frac{2y^2}{y^2+x^2}$$

$$= \frac{y^2+x^2-2y^2}{y^2+x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

Q38 If $f(x,y) = \dfrac{x^2+y^2}{x+y}$, prove that

$$(f_x - f_y)^2 = 4(1 - f_x - f_y)$$

110

Sol. Given

$$f(x,y) = \frac{x^2+y^2}{x+y}$$

Diff. partially w.r.t. x & y

$$f_x = \frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$f_x = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Now

$$f_y = \frac{(x+y).2y-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2}$$

$$f_y = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now

$$f_x - f_y = \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$= \frac{x^2+\cancel{2xy}-y^2-y^2-\cancel{2xy}+x^2}{(x+y)^2}$$

$$= \frac{2x^2-2y^2}{(x+y)^2}$$

$$= \frac{2(x-y)(x+y)}{(x+y)^2}$$

$$f_x - f_y = \frac{2(x-y)}{(x+y)}$$

Sq. both sides

$$(f_x - f_y)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

Now

Consider $1 - f_x - f_y$

$$= 1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$= \frac{(x+y)^2 - (x^2+2xy-y^2) - (y^2+2xy-x^2)}{(x+y)^2}$$

$$= \frac{\cancel{x^2}+\cancel{2xy}+y^2-\cancel{x^2}-\cancel{2xy}+\cancel{y^2}-\cancel{y^2}-2xy+x^2}{(x+y)^2}$$

$$= \frac{x^2-2xy+y^2}{(x+y)^2}$$

$$= \frac{(x-y)^2}{(x+y)^2}$$

So

$$4(1-f_x-f_y) = \frac{4(x-y)^2}{(x+y)^2}$$

Hence $(f_x - f_y)^2 = 4(1-f_x-f_y)$

Q39 Show that the function $f(x,y) = \sin(xy)$ satisfies the diff. eq.

$$x^2 f_{xx} - y^2 f_{yy} = 0$$

Sol: Given

$$f(x,y) = \sin(xy) \quad \text{——— } (1)$$

Diff (1) partially w.r.t. x

$$f_x = \cos(xy)\cdot y$$

$$f_x = y\cos(xy)$$

$$f_{xx} = y\cdot -\sin(xy)\cdot y$$

$$f_{xx} = -y^2\sin(xy)$$

Now diff. (1) partially w.r.t. y

$$f_y = \cos(xy)\cdot x$$

$$f_y = x\cos(xy)$$

$$f_{yy} = x\cdot -\sin(xy)\cdot x$$

$$f_{yy} = -x^2\sin(xy)$$

Now

Consider $x^2 f_{xx} - y^2 f_{yy}$

$$= x^2(-y^2\sin(xy)) - y^2(-x^2\sin(xy))$$

$$= -\cancel{x^2y^2\sin(xy)} + \cancel{x^2y^2\sin(xy)}$$

$$= 0$$

Q40 Let $f(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$

113

Sol. Given

$$f(x,y) = \begin{cases} x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Now

$$f_x = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right) + \tan^{-1}(y/x) \cdot 2x - y^2 \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= \cancel{x^2} \cdot \frac{x^2}{x^2+y^2} \cdot \left(\frac{-y}{\cancel{x^2}}\right) + 2x\tan^{-1}(y/x) - y^2 \cdot \frac{y^2}{y^2+x^2} \cdot \frac{1}{y}$$

$$= -\frac{x^2 y}{x^2+y^2} + 2x\tan^{-1}(y/x) - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}(y/x) - \frac{y}{(\cancel{x^2+y^2})}\left[\cancel{x^2+y^2}\right]$$

$$\boxed{f_x = 2x\tan^{-1}(y/x) - y}$$

Now

$$f_y = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - \left[\cancel{y^2} \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{\cancel{y^2}} + \tan^{-1}\frac{x}{y} \cdot 2y\right]$$

$$= \frac{x}{\frac{x^2+y^2}{x^2}} + \frac{xy^2}{y^2+x^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x}{(\cancel{x^2+y^2})}(\cancel{x^2+y^2}) - 2y\tan^{-1}\frac{x}{y}$$

$$\boxed{f_y = x - 2y\tan^{-1}\frac{x}{y}}$$

Now

$$f_{xy} = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{-h-0}{h}$$

$$= \lim_{h\to 0} (-1)$$

$$f_{xy}(0,0) = -1$$

Now

$$f_{yx} = \lim_{K\to 0} \frac{f_y(K,0) - f_y(0,0)}{K}$$

$$= \lim_{K\to 0} \frac{K-0}{K}$$

$$= \lim_{K\to 0} (1)$$

$$f_{yx}(0,0) = 1$$

So $f_{xy}(0,0) \neq f_{yx}(0,0)$

Q41 (a)

Let $f(x,y,z) = x^3 + 3yz + \sin(xyz)$

Prove that $f_{xyz} = f_{zxy}$

Sol. Given

$$f(x,y,z) = x^3 + 3yz + \sin(xyz)$$

then $f_x = 3x^2 + yz\cos(xyz)$

$$f_{xy} = z\left[y \cdot -\sin(xyz) \cdot xz + \cos(xyz) \cdot 1\right]$$

$$f_{xy} = z\cos(xyz) - xyz^2\sin(xyz)$$

$$f_{xyz} = z \cdot -\sin(xyz) \cdot xy + \cos(xyz) \cdot 1 - xy\left[z^2\cos(xyz) \cdot xy + \sin(xyz) \cdot 2z\right]$$

$$f_{xyz} = \cos(xyz) - xyz\sin(xyz) - x^2y^2z^2\cos(xyz) - 2xyz\sin(xyz)$$ 115

$$f_{xyz} = \cos(xyz) - 3xyz\sin(xyz) - x^2y^2z^2\cos(xyz) \quad \text{———} \; (A)$$

Now

$$f_z = 3y + xy\cos(xyz)$$

$$f_{zx} = y\left[x \cdot -\sin(xyz) \cdot yz + \cos(xyz) \cdot 1\right]$$

$$f_{zx} = y\cos(xyz) - xy^2z\sin(xyz)$$

$$f_{zxy} = y \cdot -\sin(xyz) \cdot xz + \cos(xyz) \cdot 1 - xz\left[y^2\cos(xyz) \cdot xz + \sin(xyz) \cdot 2y\right]$$

$$= -xyz\sin(xyz) + \cos(xyz) - x^2y^2z^2\cos(xyz) - 2xyz\sin(xyz)$$

$$f_{zxy} = \cos(xyz) - 3xyz\sin(xyz) - x^2y^2z^2\cos(xyz) \quad \text{———} \; (B)$$

From (A) & (B)

$$f_{xyz} = f_{zxy}$$

(b) If $f(x,y,z,w) = \dfrac{xy}{z+w}$, Show that

$$f_{xyzw} = \frac{2}{(z+w)^3}$$

<u>Sol:-</u> Given

$$f(x,y,z,w) = \frac{xy}{z+w}$$

Diff. partially w.r.t. x

$$f_x = \frac{y}{z+w}$$

$$f_{xy} = (f_x)_y$$

$$f_{xy} = \frac{1}{z+w}$$

$$f_{xyz} = (f_{xy})_z$$

$$f_{xyz} = \frac{-1}{(z+w)^2}$$

Now

$$f_{xyzw} = (f_{xyz})_w$$

$$= \frac{(-1)(-2)}{(z+w)^3}$$

$$= \frac{2}{(z+w)^3}$$

In problems 42–45, find $\frac{dy}{dx}$

Q42 $y^2 + x^2 y + ax^4 = 0$

Sol.

Let $f(x,y) = y^2 + x^2 y + ax^4$ —— ①

Diff. ① partially w.r.t. x & y

$$f_x = 2xy + 4ax^3$$

$$f_y = 2y + x^2$$

Now $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = -\frac{2xy+4ax^3}{2y+x^2}$$

Q43 $\quad 3x^2 - y^2 + x^3 = 0$

Sol.

Let $f(x,y) = 3x^2 - y^2 + x^3$ ——— (1)

Diff. (1) partially w.r.t. x & y

$f_x = 6x + 3x^2$

$f_y = -2y$

Now $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$$= -\frac{6x+3x^2}{-2y}$$

$$= \frac{6x+3x^2}{2y}$$

Q44 $\quad x^2 + xy + y^2 + ax + by = 0$

Sol.

Let $f(x,y) = x^2 + xy + y^2 + ax + by = 0$ ——— (1)

Diff. (1) partially w.r.t. x & y

$f_x = 2x + y + a$

$f_y = x + 2y + b$

Then $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$$= -\frac{2x+y+a}{x+2y+b} \text{ — Ans.}$$

Q45 $x^2 + xy^2 + \sin y = 0$

Sol. let $f(x,y) = x^2 + xy^2 + \sin y$ ——— (1)

Diff. (1) partially w.r.t. x & y

$$f_x = 2x + y^2$$

$$f_y = 2xy + \cos y$$

Now
$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{dy}{dx} = -\frac{2x+y^2}{2xy+\cos y}$$

End of Ch-2

Thanks to mighty God.

Differential of a function:.

Let $y = f(x)$ be a differentiable function, then its differential is defined as

$$dy = f'(x)\,dx$$

e.g.,

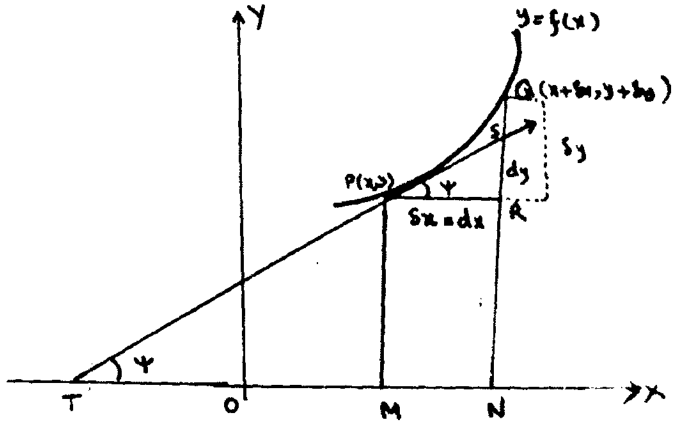
$$y = \sin^2 x$$
$$dy = 2\sin x\cos x\,dx$$

Difference b/w dy & δy:-

Let $P(x,y)$ & $Q(x+\delta x, y+\delta y)$ be any two points on $y = f(x)$.

then $\delta x = dx = PR$

$\delta y = RQ$



Now $f'(x) = \tan\psi$

$= \dfrac{RS}{PR}$

So $f'(x) = \dfrac{dy}{dx} = \dfrac{RS}{dx}$

$\Rightarrow dy = RS$

Also $\delta y = RQ$

So $\delta y \neq dy$.

Relative (average) error:- If P is the quantity to be measured & ΔP is the error in P then we define

Relative error in $P = \dfrac{\Delta P}{P}$

Percentage error in $P = \dfrac{\Delta P}{P} \times 100\ \%$

Related rate:- The rate of change of a variable with respect to time is called related rate.

EXERCISE 2.3

Find $\Delta y, dy, \Delta y - dy$ if

1. (i) $y = x^3 - 1, \quad x = 1, \quad \Delta x = -0.5$

   (ii) $y = \sqrt{3x-2}, \quad x = 2, \quad \Delta x = 0.3$

Sol.

(i) Here $y = x^3 - 1$ ———— (1)

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= [(x+\Delta x)^3 - 1] - [x^3 - 1]$$
$$= (x+\Delta x)^3 - 1 - x^3 + 1$$
$$= (x+\Delta x)^3 - x^3$$
$$= (1-0.5)^3 - (1)^3$$
$$= (0.5)^3 - 1$$
$$= 0.125 - 1$$
$$\Delta y = -0.875$$

Now from (1)

$$dy = 3x^2\,dx$$
$$dy = 3(1)^2(-0.5)$$
$$= 3(-0.5)$$
$$dy = -1.5$$

Now

$$\Delta y - dy = -0.875 - (-1.5)$$
$$= -0.875 + 1.5$$
$$= 0.625 \text{ — Ans.}$$

(ii) $y = \sqrt{3x-2}$, $x = 2$, $\Delta x = 0.3$

Sol.

Here $y = \sqrt{3x-2}$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= \sqrt{3(x+\Delta x)-2} - \sqrt{3x-2}$$
$$= \sqrt{3(2+0.3)-2} - \sqrt{3(2)-2}$$
$$= \sqrt{3(2.3)-2} - \sqrt{6-2}$$
$$= \sqrt{6.9-2} - \sqrt{4}$$
$$= \sqrt{4.9} - 2$$
$$= 2.2135 - 2$$
$$\Delta y = 0.2135$$

from (1)

$$dy = \frac{1}{2\sqrt{3x-2}} \cdot 3 \cdot dx$$
$$= \frac{3}{2\sqrt{3x-2}}\,dx$$
$$= \frac{3}{2\sqrt{3(2)-2}}(0.3)$$
$$= \frac{0.9}{2\sqrt{4}}$$
$$= \frac{0.9}{4}$$
$$dy = 0.2250$$

Now

$$\Delta y - dy = 0.2135 - 0.2250$$
$$= -0.0115 \text{ — Ans.}$$

2. Use differentials to approxiamte

(i) $\sqrt{26.2}$

Sol. We consider

$$y = f(x) = \sqrt{x} \qquad \text{———} \quad (1)$$

with $x = 25$ and $\Delta x = 1.2$

From (1), we have

$$dy = \frac{1}{2\sqrt{x}}\,dx \qquad \text{———} \quad (2)$$

Substituting $x = 25$, $dx = \Delta x = 1.2$ in (2), we get

$$dy = \frac{1}{2\sqrt{25}}(1.2) = \frac{1}{2\times 5}(1.2) = \frac{1.2}{10} = 0.12$$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$$dy = \sqrt{x+\Delta x} - \sqrt{x}$$

$$0.12 = \sqrt{25+1.2} - \sqrt{25}$$

$$0.12 = \sqrt{26.2} - 5$$

or $\sqrt{26.2} - 5 = 0.12$

$$\sqrt{26.2} = 0.12 + 5$$

$$\sqrt{26.2} = 5.12 \quad \text{— Ans.}$$

(ii) $\sqrt{80.9}$

Sol. Let $y = f(x) = \sqrt{x}$

Here $x = 81$ and $\Delta x = -0.1 = dx$

Now $dy = \frac{1}{2\sqrt{x}}dx$

$$= \frac{1}{2\sqrt{81}}(-0.1) = \frac{-0.1}{2\times 9} = \frac{-0.1}{18} = -0.00555$$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$$dy = \sqrt{x+\Delta x} - \sqrt{x}$$

$$-0.00555 = \sqrt{81-0.1} - \sqrt{81}$$

$$-0.00555 = \sqrt{80.9} - 9$$

$$\sqrt{80.9} - 9 = -0.00555$$

$$\sqrt{80.9} = -0.00555 + 9 = 8.99455$$

(iii) $\sqrt[3]{123}$

Sol.

Here $\sqrt[3]{123} = (123)^{1/3}$

Let $y = f(x) = x^{1/3}$

with $x = 125$ & $\Delta x = -2$

Now $dy = \frac{1}{3}x^{1/3-1}dx$

$= \frac{1}{3}x^{-2/3}dx$

$= \frac{1}{3x^{2/3}}dx$

$= \frac{1}{3(125)^{2/3}}(-2)$

$= \frac{-2}{3(5^3)^{2/3}}$

$= \frac{-2}{3(5)^2}$

$dy = -\frac{2}{75}$

$dy = -0.0266$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$dy = (x+\Delta x)^{1/3} - x^{1/3}$

$-0.0266 = (125-2)^{1/3} - (125)^{1/3}$

$= (123)^{1/3} - (5^3)^{1/3}$

$-0.0266 = (123)^{1/3} - 5$

$(123)^{1/3} - 5 = -0.0266$

$(123)^{1/3} = -0.0266 + 5$

$\sqrt[3]{123} = 4.9734$

(iv) $\cos 61^\circ$

Sol. Let $y = f(x) = \cos x$

with $x = 60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$

& $\Delta x = 1^\circ = 1 \cdot \frac{\pi}{180} = \frac{\pi}{180}$

Now $dy = -\sin x\, dx$

$= -\sin(\pi/3) \cdot \frac{\pi}{180}$

$= -\frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}$

$dy = -\frac{\sqrt{3}\pi}{360}$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$dy = \cos(60^\circ + 1^\circ) - \cos 60^\circ$

$-\frac{\sqrt{3}\pi}{360} = \cos 61^\circ - \frac{1}{2}$

$-\frac{(1.732)(3.14)}{360} = \cos 61^\circ - 0.5$

$-0.1512 = \cos 61^\circ - 0.5$

$\cos 61^\circ - 0.5 = -0.1512$

$\cos 61^\circ = -0.1512 + 0.5$

$= 0.4848$

(v) $(3.02)^4$

Sol. Let

$$y = f(x) = x^4 \quad \text{with}$$
$$x = 3 \text{ and } \Delta x = 0.02$$
$$dy = 4x^3\,dx$$
$$= 4 \times 3^3\,(0.02)$$
$$dy = 2.16$$

Since

$$dy \approx \Delta y = f(x+\Delta x) - f(x)$$
$$dy = (x+\Delta x)^4 - x^4$$
$$= (3+0.02)^4 - (3)^4$$
$$2.16 = (3.02)^4 - 81$$
$$(3.02)^4 - 81 = 2.16$$
$$(3.02)^4 = 2.16 + 81 = 83.16$$

(vi) $\tan 29°$

Sol.

Let $y = f(x) = \tan x$

with $x = 30° = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

& $\Delta x = -1° = -1 \cdot \frac{\pi}{180} = -\frac{\pi}{180}$

Now $dy = \sec^2 x\,dx$

$$= \sec^2(\pi/6) \cdot \left(-\frac{\pi}{180}\right)$$
$$= \frac{1}{\cos^2 \pi/6} \cdot \frac{-3.14}{180}$$
$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} \cdot \frac{-3.14}{180}$$
$$= -\frac{4}{3} \times \frac{3.14}{180}$$
$$dy = -0.0233$$

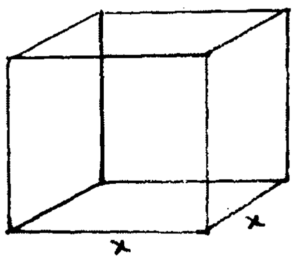
Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$$dy = \tan(x+\Delta x) - \tan x$$
$$-0.0233 = \tan(30° - 1°) - \tan 30°$$
$$-0.0233 = \tan 29° - \frac{1}{\sqrt{3}}$$
$$\tan 29° - \frac{1}{\sqrt{3}} = -0.0233$$
$$\tan 29° - 0.5733 = -0.0233$$
$$\tan 29° = -0.0233 + 0.5733$$
$$\tan 29° = 0.55$$

3. **The side of a cube is measured with a possible error of ± 2 %. Find the percentage error in the surface area of one face of the cube.**

Sol. Let x be edge of the cube

then Area A of a face is



$$A = x.x = x^2$$

$$dA = 2x\,dx$$

$$\text{Relative error} = \frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x}$$

But $\dfrac{dx}{x} = \pm 0.02$

Therefore,

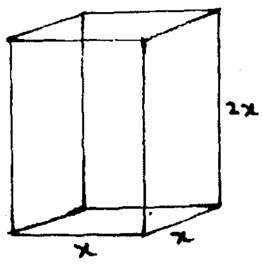
$$\frac{dA}{A} = 2(\pm 0.02)$$

$$= \pm 0.04$$

The percentage error in the surface area is $= \frac{dA}{A}\times 100 = \pm 0.04\times 100 = \pm 4\%$.

4. **A box with a square base has its height twice its width. if the width of the box is 8.5 inches (in.) with a possible error of ± 0.3 in, find the possible error in the volume of the box.**

Sol. Let x in be the width of the box. Then its volume V is



$$V = L.W.h = x.x.2x = 2x^3$$

$$dV = 6x^2\,dx$$

But $\quad dx = \pm 0.3$

Therefore change in volume is

$$dV = 6(8.5)^2(\pm 0.3)$$

$$= \pm (6)(72.25)(0.3)$$

$$= \pm 130.05 \text{ Cubic inches.}$$

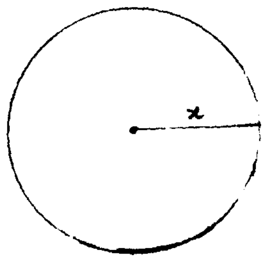
5. The radius x of a circle increases from $x = 10$ cm to $x = 10.1$ cm. Find the corresponding change in the area of the circle. Also find the percentage change in the area.

Sol. Let A be area of the circle of radius x. Then

$$A = \pi x^2$$
$$\Rightarrow \quad dA = 2\pi x\, dx$$

Now, $x = 10$ cm and $\Delta x = dx = 0.1$ cm

Change in the area of the circle is



$$dA = 2\pi(10)(0.1) = 2\pi \text{ cm}^2$$

Relative change in the area is

$$\frac{\Delta A}{A} = \frac{2\pi}{\pi(10)^2} = \frac{2}{100} = 0.02$$

Percentage change $= \frac{2}{100} \times 100 = 2\%$.

6. The diameter of a tree was 8 inches. After one year the circumference of the tree increased by 2 inches. How much did

(i) the diameter of the tree increase?

(ii) the cross-section area of the tree change?

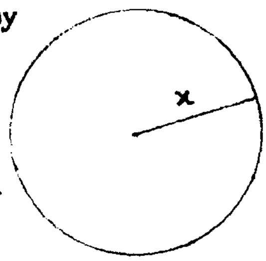
Sol. If x is the radius of the tree, then its circumference

$$C = 2\pi x$$

Therefore, $dC = 2\pi\, dx$

Change in circumference is $dC = 2$

and so the change Δx in radius is given by



$$2 = 2\pi\, dx$$

or $\quad dx = \frac{1}{\pi}$

Thus the diameter increased by $\frac{2}{\pi}$ inches.

Area A of the cross-section of tree is

$$A = \pi x^2$$
$$\Rightarrow \quad dA = 2\pi x\, dx$$

When $x = 4$, $dx = \frac{1}{\pi}$ and change in area is

$$dA = 2\pi \; 4 \cdot \frac{1}{\pi} = 8 \text{ sq. inches}$$

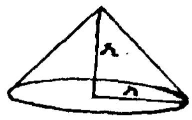
7. **Sand pouring from a chute forms a conical pile whose altitude is always equal to the radius. If the radius of the pile is 10 cm, find the approximate change in radius, when volume increases by 2 cm^3.**

Sol. The volume V of the conical pile of radius r and height r is

$$V = \frac{1}{3}\pi r^3$$

$$\Rightarrow dV = \frac{1}{3}\pi \cdot 3r^2 dr$$

$$\text{or } dV = \pi r^2 dr \quad \text{——} (1)$$



Now given that $\Delta V \approx dV = 2\text{cm}^3$ when $r = 10$

So req. change in radius $= \Delta r = dr = ?$

Hence from (1) $\quad 2 = \pi(10)^2 dr$

$$2 = 100\pi dr \Rightarrow dr = \frac{2}{100\pi} = \frac{1}{50\pi}\text{cm.}$$

So change in radius of pile $= \frac{1}{50\pi}$cm.

8. **A dome is in the shape of a hemisphere with radius 60 ft. The dome is to be painted with a layer of 0.01 inch thickness. Use differentials to estimate the amount of the paint required.**

Sol. If V is volume of hemisphere with radius r, then

$$V = \frac{2\pi r^3}{3}$$

$$\Rightarrow dV = \frac{2}{3}\pi \cdot 3r^2 dr$$

$$dV = 2\pi r^2 dr \quad \text{——} (1)$$

we want to find dV

when $r = 60\text{ft.}$ & $dr = 0.01''$

$$= \frac{0.01}{12}\text{ft}$$

$$= \frac{1}{12 \times 100}\text{ft}$$

So $dr = \frac{1}{1200}$ ft.

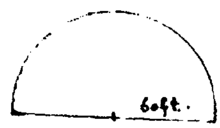
Putting values in (1)

$$dV = 2\pi(60)^2 \cdot \frac{1}{1200}$$

$$= 2\pi \cdot 3600 \cdot \frac{1}{1200}$$

$$= 2\pi \times 3$$

$$= 6\pi\,\text{ft}^3$$



9. **The side of a building is in the shape of a square surmounted by an equilateral triangle. If the length of the base is 15 m with an error of 1 %, find the percentage error in the area of the side.**

Sol. Let x m be the length of the base. Then area A of the side is given by

A = area of square + area of triangle

$$= x^2 + \frac{1}{2}\cdot x \cdot h$$

$$= x^2 + \frac{x}{2}\cdot\frac{\sqrt{3}x}{2}$$

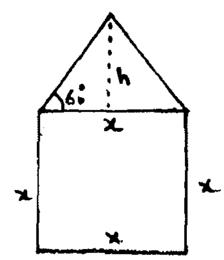
$$A = x^2 + \frac{\sqrt{3}}{4}x^2$$

$$\Rightarrow dA = \left(2x + \frac{\sqrt{3}}{4}\cdot 2x\right)dx$$

$$\because \frac{h}{\frac{x}{2}} = \tan 60^\circ$$

$$\frac{2h}{x} = \sqrt{3}$$

$$h = \frac{\sqrt{3}x}{2}$$



$dA = \left(2x + \frac{\sqrt{3}x}{2}\right)dx$ ——(1)

Now given that $\frac{dx}{x} = 0.01$

Now percentage error in $A = ?$ when $x = 15$

i.e., $\frac{dA}{A} \times 100 = ?$

Now from (1)

$$\frac{dA}{A} = \frac{\left(2x + \frac{\sqrt{3}x}{2}\right)dx}{x^2 + \frac{\sqrt{3}}{4}x^2} = \left[\frac{2x + \frac{\sqrt{3}}{2}x}{x + \frac{\sqrt{3}}{4}x}\right]\frac{dx}{x}$$

$$\frac{dA}{A} = \left[\frac{2(15) + \frac{\sqrt{3}}{2}(15)}{15 + \frac{\sqrt{3}}{4}(15)}\right] \times 0.01$$

$$= \left[\frac{30 + \frac{\sqrt{3}\cdot 15}{2}}{15 + \frac{\sqrt{3}\cdot 15}{4}}\right] \times \frac{1}{100}$$

$$= \left(\frac{120 + 30\sqrt{3}}{60 + 15\sqrt{3}}\right) \times \frac{1}{100}$$

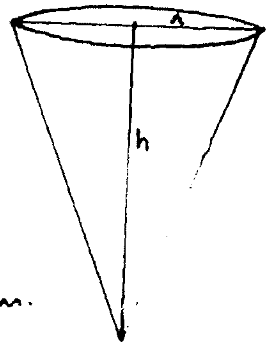
$$= 2 \times \frac{1}{100} = \frac{1}{50}$$

So % error in area

$$= \frac{dA}{A} \times 100 = \frac{1}{50} \times 100 = 2\ \%.$$

10. A boy makes a paper cup in the shape of a right circular cone with height four times its radius. If the radius is changed from 2 cm to 1.5 cm but the height remains four times the radius, find the approxiamte decrease in the capacity of the cup.

Sol. If r is the radius of the base and h is height of the cup, then its volume V is given by



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (4r)$$

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow dV = \frac{4}{3}\cdot\pi\cdot 3r^2 dr$$

$$\text{So } dV = 4\pi r^2 dr \text{ ——(1)}$$

As it is given $r = 2\text{cm.}$ & $dr = 1.5 - 2 = -0.5\text{cm.}$

So from (1) $dV = 4\pi(2)^2(-0.5)$

$$= 4\pi \cdot 4 \cdot \frac{-1}{2}$$

$$= 4\pi(-2)$$

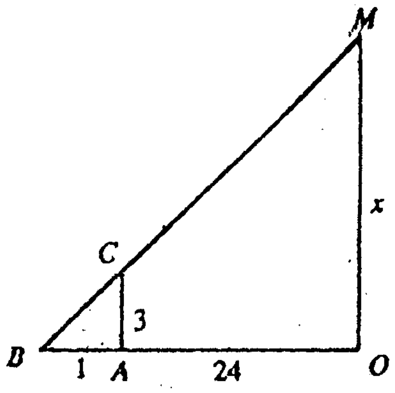
$$dV = -8\pi$$

is the req. change in Capacity of Cup.

The $-$ve sign shows that there is decrease in the capacity of the cup.

11. To estimate the height of Minar-i-Pakistan, the shadow of a 3 m pole placed 24 m from the Minar is measured. If the length of the shadow is 1 m with a percentage error of 1 %, find the height of the Minar. Also find the percentage error in the height so found.

Sol. Let OM be the minar & AC be the pole.



If x m is height of the Minar, then from the figure

$$\frac{x}{25} = \frac{3}{1}$$

Therefore, $x = 25 \times 3 = 75$.

Height of the minar $= 75$ m.

If y is the actual length of the shadow of the pole, then

$$\frac{y+24}{x} = \frac{y}{3}$$

or $\quad 3y + 72 = xy$

or $\quad 3dy = x\,dy + y\,dx$

or $\quad (3-x)\,dy = y\,dx$

or $\quad (3-x)\dfrac{dy}{y} = dx \quad$ —— (1)

Now $\dfrac{dy}{y} = \pm\, 0.01$. When $x = 75$, relative error in the height $= \dfrac{dx}{x}$.

Now from (1) $\quad dx = (3-x).\dfrac{dy}{y}$

$= (3-75)(\pm 0.01)$

$= (-72)(\pm 0.01)$

So $\dfrac{dx}{x} = \dfrac{(-72)(\pm 0.01)}{75}$

or $\dfrac{dx}{x} = \pm 0.0096$

So Percentage error in height $= \dfrac{dx}{x} \times 100$

$= \pm 0.0096 \times 100$

$= \pm 0.96\%$.

12. Oil spilled from a tanker spreads in a circle whose radius increases at the rate of 2 ft/sec. How fast is the area increasing when the radius of the circle is 40 ft?

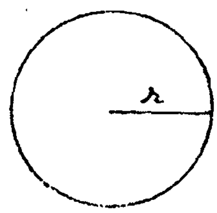
Sol. Let r be the radius of the circle at any instant t. Then area A of the circle is

$$A = \pi r^2$$

Diff. w.r.t. t

$$\frac{dA}{dt} = \pi . 2r . \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$



We have to find $\frac{dA}{dt}$ when $\frac{dr}{dt} = 2$ and $r = 40$. Substituting into (1), we have

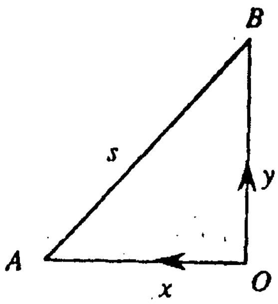
$$\frac{dA}{dt} = 2\pi \times 40 \times 2$$

$$\Rightarrow 160\pi$$

Thus area of the circle changes at the rate of 160π ft^2/sec.

13. **From a point O, two cars leave at the same time. One car travels west and after t sec. its position is $x = t^2 + t$ ft. The other car travels north and it covers $y = t^2 + 3t$ ft. in t sec. At what rate is the distance between the two cars changing after 5 sec?**

Sol. Let A, B be the positions of the two cars at any instant t and let s be the distance between them at this instant.



$$s^2 = x^2 + y^2 \qquad (1)$$

diff. w.r.t. t

$$2s\frac{ds}{dt} = 2x.\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow s\,dy/dt = x\frac{dx}{dt} + y\frac{dy}{dt} \qquad (2)$$

We have to find $\frac{ds}{dt}$ at the instant when $t = 5$.

We have

$$x = t^2 + t \qquad (3)$$

$$y = t^2 + 3t \qquad (4)$$

Differentating (3) and (4) w.r.t. t, we have

$$\frac{dx}{dt} = 2t + 1,$$

$$\frac{dy}{dt} = 2t + 3,$$

& at $t = 5$, $\frac{dx}{dt} = 11$ & $\frac{dy}{dt} = 13$

After 5 sec, the distances of the two cars from O are

$$x = 5^2 + 5 = 30$$

$$y = 5^2 + 15 = 40$$

& from (1) $\quad s^2 = 30^2 + 40^2$

$$= 900 + 1600$$

$$s^2 = 2500$$

So $S = 50$

Putting values in eq. (2)

$$50\frac{ds}{dt} = 30\times 11 + 40\times 13$$

or $$50\frac{ds}{dt} = 330 + 520$$

$$\frac{ds}{dt} = \frac{850}{50}$$

or $$\frac{ds}{dt} = 17$$

Therefore, the distance between the two cars is changing at the rate of 17 ft./sec.

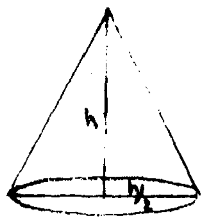
14. Sand falls from a container at the rate of 10 ft^3/min and forms a conical pile whose height is always double the radius of the base. How fast is the height increasing when the pile is 5 ft high?

Sol. Let h be the height of the pile at any instant t. Radius of the pile $= \frac{h}{2}$. Volume V of the pile is

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 . h$$

$$= \frac{1}{3}\pi . \frac{h^3}{4}$$

$$V = \frac{\pi}{12}h^3$$



Now $$\frac{dV}{dt} = \frac{\pi}{12}.3h^2.\frac{dh}{dt}$$

or $$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt} \quad \text{——— (1)}$$

It is given that $\frac{dV}{dt} = 10$ & we want to find $\frac{dh}{dt}$ when $h = 5$

Putting values in (1)

$$10 = \frac{\pi}{4}(5)^2 . \frac{dh}{dt}$$

$$40 = 25\pi . \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{40}{25\pi}$$

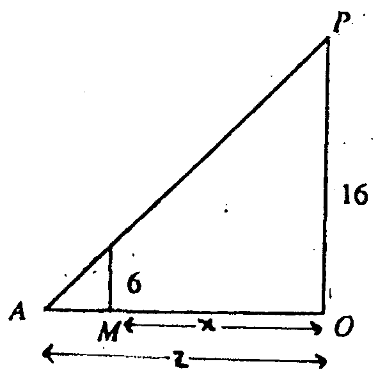
$$\frac{dh}{dt} = \frac{8}{5\pi} = \frac{8}{5\times 3.14} = 0.51$$

So the height of pile is changing at the rate of 0.51 ft/minute.

15. A 6 ft tall man is walking toward a lamp post 16 ft high at a speed of 5 ft/sec. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing?

Sol. Let x be man's distance from the lamp post OP and z the distance of the tip of his shadow from O.

i.e. $OM = x$, $OA = z$



From the similar triangles, we have

$$\frac{16}{z} = \frac{6}{z-x}$$

$$16z - 16x = 6z$$

$$16z - 6z = 16x$$

$$10z = 16x$$

$$5z = 8x$$

$$5\frac{dz}{dt} = 8\frac{dx}{dt}$$

It is given that $\frac{dx}{dt} = 5$

$$5.5\frac{dz}{dt} = 8\times5$$

$$\frac{dz}{dt} = 8$$

Therefore the tip of man's shadow is moving at the rate of 8 ft./sec.

If y is the length of the shadow then $MA = y$. From the similar triangles we have

$$\frac{16}{x+y} = \frac{6}{y}$$

i.e., $8y = 5x$

Therefore $8\frac{dy}{dt} = 5\frac{dx}{dt}$.

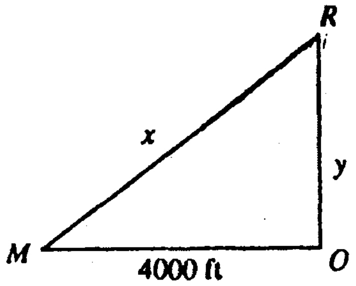
Substituting $\frac{dx}{dt} = 5$, we find that

$$\frac{dy}{dt} = \frac{25}{8}$$

Thus the shadow is changing at the rate of $\frac{25}{8}$ ft/sec.

16. At a distance of 4000 ft from a launching site, a man is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 ft/sec. when it is at an altitude of 3000ft, how fast is the distance between the rocket and the man changing at this instant?

Sol. Let y be altitude of the rocket and x be the distance between the man and the rocket at any instant t.



We have

$$x^2 = y^2 + 4000^2 \qquad (1)$$

When $y = 3000$ ft,

we have from (1),

$$x^2 = 3000^2 + 4000^2$$
$$= 9000000 + 16000000$$
$$x^2 = 25000000$$
$$\Rightarrow x = 5000$$

Diff. (1) w.r.t. t

$$2x\frac{dx}{dt} = 2y\cdot\frac{dy}{dt}$$
$$x\frac{dx}{dt} = y\frac{dy}{dt}$$

It is given that $\frac{dy}{dt} = 600$ when $y = 3000$

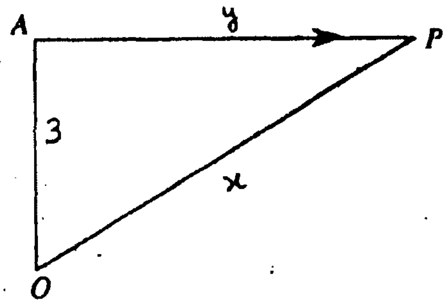
So $5000\cdot\frac{dx}{dt} = 3000 \times 600$

$$\frac{dx}{dt} = \frac{3000\times 600}{5000} = \frac{3}{5}\times 600 = 3\times 120 = 360$$

Thus the distance between the rocket and the man is changing at the rate of 360 ft/sec.

17. An airplane flying horizontally at an altitude of 3 miles and a speed of 480 miles/hr. passes directly above an observer on the ground. How fast is the distance of the observer to the airplane increasing after 30 sec?

Sol. Let O be the observer on the ground and P be the plane at some instant t. Let



$$OP = x, \quad AP = y$$

It is given that $OA = 3$

From the right triangle, we have

$$3^2 + y^2 = x^2 \text{ ——— } (1)$$

The distnace travelled by the plane 30 sec. after it has passed above the observer $= \frac{480}{60\times 60}\times 30 = 4$ miles. Put in (1)

$$9 + (4)^2 = x^2$$
$$25 = x^2$$
$$x = 5$$

We have to find $\frac{dx}{dt}$ at the instant when $t = 30$ sec. and $\frac{dy}{dt} = 480$.

Now diff. (1) w.r.t. t

$$2y.\frac{dy}{dt} = 2x.\frac{dx}{dt}$$

$$y\frac{dy}{dt} = x\frac{dx}{dt}$$

$$x\frac{dx}{dt} = y.\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{y}{x}.\frac{dy}{dt}$$

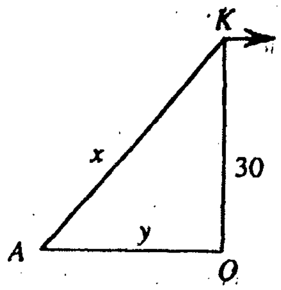
So at $t = 30$, $\frac{dx}{dt} = \frac{4}{5} \times 480$

$$= 384$$

The rate of change of distance of the plane from the observer = 384 miles/hr.

18. A boy flies a kite at an altitude of 30 m. If the kite flies horizontally away from the boy at the rate of 2 m/sec, how fast is the string being let out when the length of the string released is 70 m?

Sol. Let x be the length of the string let out at some instant t, K be the kite at an altitude of 30 m and let $AO = y$. The kite flies horizontally away from the boy at the rate of 2 m/sec.



From ΔAOK, we have

$$x^2 = 30^2 + y^2 \quad \text{———— (1)}$$

Diff. (1) w.r.t. t

$$2x.\frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$x.\frac{dx}{dt} = y.\frac{dy}{dt} \quad \text{———— (2)}$$

When $x = 70$ then from (1)

$$(70)^2 = (30)^2 + y^2$$

$$4900 = 900 + y^2$$

$$4900 - 900 = y^2$$

$$4000 = y^2$$

$$y = \sqrt{400 \times 10}$$

$$y = 20\sqrt{10}$$

At this time $\frac{dy}{dt} = 2$

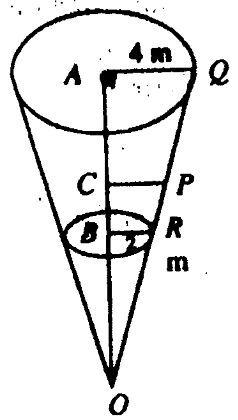
So from (2)

$$70.\frac{dx}{dt} = 20\sqrt{10} \times 2$$

$$\frac{dx}{dt} = \frac{40\sqrt{10}}{70} = \frac{4\sqrt{10}}{7}$$

Hence the string is being let out at the rate of $\frac{4\sqrt{10}}{7}$ m/sec.

19. A water tank is in the shape of frustrum of a cone with height 6 m and upper and lower radii 4 m and 2 m, respectively. If water pours into the tank at the rate of 20 m^3/min, how fast is the water level rising when the water is half way up?



Sol. Extend the tank downward so as to form a cone. Let $BO = x$ m so that the height of the cone is $x + 6$.

Suppose that at some instant water level is at C where $BC = y$ and let $CP = r$.

From similar Δ's AOQ and BOR, we get

$$\frac{6+x}{4} = \frac{x}{2}$$

$$4x = 12 + 2x$$

$$4x - 2x = 12$$

$$2x = 12$$

$$x = 6$$

So $\boxed{BO = 6}$

Again from similar Δs COP & BOR

$$\frac{y+6}{r} = \frac{6}{2}$$

$$6r = 2y + 12$$

$$r = \frac{2(y+6)}{6}$$

$$r = \frac{y+6}{3}$$

Now the volume of frustum with upper radius r & lower radius 2 is

$$V = \frac{1}{3}\pi r^2(y+6) - \frac{1}{3}\pi(2)^2 \times 6$$

$$= \frac{\pi}{3}\cdot\frac{(y+6)^2}{9}\cdot(y+6) - \frac{\pi}{3}\times 24$$

$$V = \frac{\pi}{27}(y+6)^3 - 8\pi$$

Diff. w.r.t. t

$$\frac{dv}{dt} = \frac{\pi}{27}\cdot 3(y+6)^2\cdot\frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{\pi}{9}\times(y+6)^2\cdot\frac{dy}{dt}$$

As it is given that $\frac{dv}{dt} = 20$

& we want to find $\frac{dy}{dt}$ when water is half way up

i.e., when $y = 3$

Hence from above eq.

$$20 = \frac{\pi}{9}(3+6)^2\cdot\frac{dy}{dt}$$

$$20 = \frac{\pi}{9}\times 81\cdot\frac{dy}{dt}$$

$$\frac{20\times 9}{81\pi} = \frac{dy}{dt}$$

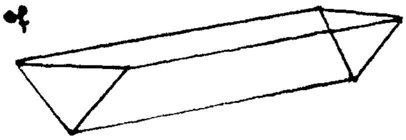
$$\frac{dy}{dt} = \frac{20}{9\pi}$$

Hence water level is rising at the rate of

$\frac{20}{9\pi}$ m/min.

20. A 12 m long water trough, with vertical cross-sections in the shape of equilateral triangles (one vertex down) is being filled at the rate of $4m^3$/min. How fast is the water level rising at the instant when the depth of the water is $1\frac{1}{2}$ m?

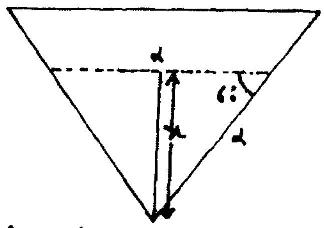
Sol. Suppose the water is x feet deep
then area of vertical cross-section of water is



$$A = \frac{x^2}{2\sin 60^\circ}$$

$$= \frac{x^2}{2\times\frac{\sqrt{3}}{2}}$$

$$A = \frac{x^2}{\sqrt{3}}$$



Area of vertical cross-section of water is

$$A = \frac{1}{2}\cdot d\cdot x$$

But $\frac{x}{d} = \sin 60^\circ$

$$\Rightarrow \frac{d}{x} = \frac{1}{\sin 60^\circ}$$

$$d = x\operatorname{cosec} 60^\circ$$

So $A = \frac{1}{2}\cdot x\operatorname{cosec}60^\circ\cdot x = \frac{x^2}{2\sin 60^\circ}$

then Volume of water at this time is

$$V = \frac{x^2}{\sqrt{3}}\times 12$$

$$V = \frac{12x^2}{\sqrt{3}}$$

Diff. w.r.t. t

$$\frac{dV}{dt} = \frac{24x}{\sqrt{3}}\cdot\frac{dx}{dt}$$

We want to find $\frac{dx}{dt}$ at the time when $x = \frac{3}{2}$ & $\frac{dV}{dt} = 4$

So Putting values

$$4 = \frac{24}{\sqrt{3}}\times\frac{3}{2}\times\frac{dx}{dt}$$

$$4 = 12\sqrt{3}\cdot\frac{dx}{dt}$$

$$\frac{4}{12\sqrt{3}} = \frac{dx}{dt}$$

or $\frac{dx}{dt} = \frac{1}{3\sqrt{3}}$

So the water level is rising at the rate of $\frac{1}{3\sqrt{3}}$ m/min.

Mathematics - A Course
Class : B.Sc Part - I
Book: Calculus with Analytic Geometry

Chapter No: 02
The Derivative
Page : 01

Newton - Raphson Method:- Exercise 2.4

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use the Newton-Raphson method to approximate, up to four Places of decimal, a root of each of the following:-

1. $x^3 - 3x - 3 = 0$ with $x_0 = 2$

Sol:-

$x^3 - 3x - 3 = 0 \longrightarrow$ eq (i).

Then differentiate eq (i) w.r.t "x".

$3x^2 - 3$ eq (ii).

Now By using Newton-Raphson Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put $n = 0$:-

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{[-1]}{9}$$

$$x_1 = 2 + \frac{1}{9}$$

$$\boxed{x_1 = 2.1111}$$

∴ The value of $x_0 = 2$ given in question.

∴ By Putting the value of $x_0 = 2$ in eq (i) we get $= -1$

∴ By Putting the value of $x_0 = 2$ in eq (ii) we get 9

At $n = 1$:-

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.1111 - \frac{0.0753}{10.3702}$$

$$x_2 = 2.1111 - 0.0072$$

$$\boxed{x_2 = 2.1039}$$

At $n = 2$:-

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.1039 - \frac{0.0009}{10.2791}$$

$$\boxed{x_3 = 2.1039}$$

As the root "2.1039" repeated two times so the Required root is $\boxed{2.1039}$ Ans.

$$\boxed{x = 2.1039} \text{ Ans.}$$

Exercise set 2.4

2. $x^3 - 5x + 3 = 0$ with $x_0 = 0$

Soln.

By using Newton's Raphson method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x^3 - 5x + 3 = 0 \longrightarrow$ eq. (i)

$3x^2 - 5 \longrightarrow$ eq (ii)

Put $n = 0$:-

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

∴ By Putting the value of x_0 in eq (i) we get 3

∴ By Putting the value of x_0 in eq (ii) we get -5

$$x_1 = 0 - \frac{3}{-5}$$

$$x_1 = 0 + \frac{3}{5}$$

$$\boxed{x_1 = 0.6000}$$

Put $n = 1$ in eq (i)

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.6000 - \frac{0.2160}{-3.9200}$$

$$x_2 = 0.6000 + \frac{0.2160}{3.9200}$$

$$x_2 = 0.6000 + 0.0551$$

$$\boxed{x_2 = 0.6551}$$

Put $n = 2$ in eq (i):-

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.6551 - \frac{0.0056}{-3.7125}$$

$$x_3 = 0.6551 + \frac{0.0056}{3.7125}$$

$$x_3 = 0.6551 + 0.0015$$

$$\boxed{x_3 = 0.6566}$$

Put $n = 3$ in eq (i)

$$x_{3+1} = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 0.6566 - \frac{0}{-3.7066}$$

$$x_4 = 0.6566 + 0$$

$$\boxed{x_4 = 0.6566}$$

Since the root 0.6566 is repeated two times so reqvired root is $\boxed{x = 0.6566}$

$\boxed{x = 0.6566}$ Ans.

Exercise 2.4

3. $e^{-x} - \sin x = 0$ with $x_0 = 0.5$

Sol:-

By using Newton's Raphson Method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$e^{-x} - \sin x = 0 \rightarrow$ eq (i)

$-e^{-x} - \cos x \rightarrow$ eq (ii)

Put $n=0$ in eq (i) :-

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{0.1269}{-1.4840}$$

$$x_1 = 0.5 + 0.0855$$

$$\boxed{x_1 = 0.5855}$$

Put $n=1$ in eq (i) :-

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.5855 - \frac{0.0041}{-1.3902}$$

$$x_2 = 0.5855 + 0.0029$$

$$\boxed{x_2 = 0.5884}$$

Put $n=2$ in eq (i) :-

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.5884 - \frac{0.0002}{-1.3870}$$

$$x_3 = 0.5884 + 0.0001$$

$$\boxed{x_3 = 0.5885}$$

Put $n=3$ in eq (i)

$$x_{3+1} = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 0.5885 - \frac{0}{-1.3868}$$

$$x_4 = 0.5885 + 0$$

$$\boxed{x_4 = 0.5885}$$

Since the root 0.5885 is repeated two times so the required root is $\boxed{x = 0.5885}$

$$\boxed{x = 0.5885}$$ Ans.

Mathematics-A course
Class: B.Sc part 1
Book: Calculus with Analytic Geometry

Chapter No: 02
The Derivative
Page: 04

Exercise 2.4

4. $e^x - 3x = 0$ with $x_0 = 0$

Sol:-

By using Newton's Raphson method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$e^x - 3x = 0 \rightarrow$ eq (i).

$e^x - 3 \rightarrow$ eq (ii).

Put $n=0$ in eq (i).

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{1}{-2}$$

$$x_1 = 0 + 0.5000$$

$$\boxed{x_1 = 0.5000}$$

Put $n=1$ in eq (i).

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.5000 - \frac{0.1487}{-1.3513}$$

$$x_2 = 0.5000 + 0.1100$$

$$\boxed{x_2 = 0.6100}$$

Put $n=2$ in eq (i) :-

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.6100 - \frac{0.0104}{-1.1596}$$

$$x_3 = 0.6100 + 0.0089$$

$$\boxed{x_3 = 0.6189}$$

Put $n=3$ in eq (i):

$$x_{3+1} = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 0.6189 - \frac{0.0002}{-1.1437}$$

$$x_4 = 0.6189 + 0.0001$$

$$\boxed{x_4 = 0.6190}$$

Put $n=4$ in eq (i).

$$x_{4+1} = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$x_5 = 0.6190 - \frac{0}{-1.1433}$$

$$x_5 = 0.6190 + 0$$

$$\boxed{x_5 = 0.6190}$$

Since By the following "0.6190" repeated two times so the the required root is 0.6190.

$$\boxed{x = 0.6190} \text{ Ans.}$$

Exercise 2.4

5.) $4\sin x = e^x$ in the intervals $]0, 0.5[$

Sol:-

$f(x) = 4\sin x = e^x \rightarrow eq(i)$

$eq(i) \Rightarrow f(x) = 4\sin x - e^x \rightarrow eq(ii)$.

Now in the question we have to find the value x_0.
Then put the value of the intervals $]0, 0.5[$ in eq(ii).

$f(0) = 4\sin(0) - e^0$

$\therefore e^0 = 1$

$f(0) = 0 - 1$

$\therefore \sin 0° = 0$

$\boxed{f(0) = -1}$

Now at interval 0.5 put in eq (2)

$f(0.5) = 4\sin(0.5) - e^{0.5}$

$= 4(0.4794) - 1.6487$

$= 1.9176 - 1.6487$

$\boxed{f(0.5) = 0.2689}$

$\boxed{x_0 = 0.2618}$

$\therefore$ The value of $\boxed{x_0 = 0.2618}$

$f'(x) = 4\sin x - e^x \rightarrow eq(ii)$

Differentiating eq(ii) w.r.t "x"

$= 4\cos x - e^x$

Now By using Newton's Raphson method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put $n = 0$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.2618 - \frac{(-0.264)}{2.5640}$$

$$= 0.2618 + \frac{0.264}{2.5640}$$

$$= 0.2618 + 0.1029$$

$$\boxed{x_1 = 0.3647}$$

Put n = 1 in (i)

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.3647 - \frac{(-0.0134)}{2.2968}$$

$$x_2 = 0.3647 + \frac{0.0134}{2.2968}$$

$$\boxed{x_2 = 0.3705}$$

Put n = 2 in (i)

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.3705 - \frac{(-0.0001)}{2.2802}$$

$$x_3 = 0.3705 + 0.0000$$

$$\boxed{x_3 = 0.3705}$$

Hence the required root is 0.3705 because it repeated two times in the ~~eq~~ eq.

$$\boxed{x = 0.3705}$$ Ans.

Mathematics - A Course
Book: Calculus with Analytic Geometry
Class: B.Sc. Part I

Chapter No: 02
The Derivative

Exercise 2.4

6). $\sin x = 1 - x$ with $x_0 = 0$.

Sol:-

By using Newton-Raphson method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = 1 - x - \sin x = 0 \rightarrow eq(i)$.

$f'(x) = -1 - \cos x \rightarrow eq(ii)$.

Put $n = 0$ in eq(i):-

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{1}{-2}$$

$x_1 = 0 + \frac{1}{2} \Rightarrow \boxed{x_1 = 0.5000}$

Put $n = 1$ in eq(i).

$$x_{1+1} = x_n - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.5000 - \frac{(-0.0206)}{1.8776}$$

$= 0.5000 + 0.0109$

$\boxed{x_2 = 0.5110}$

Put $n = 2$ in eq(i):-

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.5110 - \frac{0}{1.8723}$$

$x_3 = 0.5110 - 0$

$\boxed{x_3 = 0.5110}$

As "0.5110" is repeated two times so this is a required root.

$\boxed{x = 0.5110}$ Ans. (The End).

Higher derivatives:-

Let $y = f(x)$

then its first, second, third, ... nth derivatives are denoted by

$\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ..., $\frac{d^ny}{dx^n}$

y_1 , y_2 , y_3 , ..., y_n

y' , y'' , y''' , ..., y^n

$f'(x)$, $f''(x)$, $f'''(x)$, ..., $f^n(x)$

e.g., Let $y = x^4 + 2x^3 + 3x^2 + 7x + 5$

Diff. w.r.t. x successively

$y_1 = 4x^3 + 6x^2 + 6x + 7$

$y_2 = 12x^2 + 12x + 6$

$y_3 = 24x + 12$

$y_4 = 24$

Derivatives found above are called higher derivatives.

Some standard nth derivatives:-

① nth derivative of e^{ax}:-

Let $y = e^{ax}$

Diff. w.r.t. x successively

$y_1 = e^{ax} \cdot a = ae^{ax}$

$y_2 = ae^{ax} \cdot a = a^2 e^{ax}$

$y_3 = a^2 e^{ax} \cdot a = a^3 e^{ax}$

$y_4 = a^3 e^{ax} \cdot a = a^4 e^{ax}$

$\boxed{y_n = a^n e^{ax}}$

48

② **nth derivative of $(ax+b)^m$:-**

Let $y = (ax+b)^m$

Diff. w.r.t. x successively

$y_1 = m(ax+b)^{m-1} \cdot a = ma(ax+b)^{m-1}$

$y_2 = ma(m-1)(ax+b)^{m-2} \cdot a = m(m-1) \cdot a^2 (ax+b)^{m-2}$

$y_3 = m(m-1)a^2(m-2)(ax+b)^{m-3} \cdot a = m(m-1)(m-2) \cdot a^3 (ax+b)^{m-3}$

———

———

———

$y_n = m(m-1)(m-2)\ldots\ldots(m-(n-1)) \cdot a^n (ax+b)^{m-n}$

$= m(m-1)(m-2)\text{------}(m-n+1) \cdot (ax+b)^{m-n} \cdot a^n$

$= \dfrac{m(m-1)(m-2)\text{------}(m-n+1)(m-n)(m-n-1)\ldots 3 \cdot 2 \cdot 1 \cdot (ax+b)^{m-n} \cdot a^n}{(m-n)(m-n-1)\text{------}3 \cdot 2 \cdot 1}$

$= \dfrac{m! \cdot (ax+b)^{m-n} \cdot a^n}{(m-n)!}$

So $\boxed{y_n = \dfrac{m! \cdot a^n (ax+b)^{m-n}}{(m-n)!}}$

③ **nth derivative of $\dfrac{1}{ax+b}$:-**

Let $y = \dfrac{1}{ax+b}$

or $y = (ax+b)^{-1}$

Diff. w.r.t. x successively

$y_1 = (-1)(ax+b)^{-2} \cdot a$

$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a \cdot a = (-1)(-2)(ax+b)^{-3} \cdot a^2$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^2 \cdot a = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^3$$ 49

$$y_n = (-1)(-2)(-3)\cdots\cdots(-n)(ax+b)^{-(n+1)} \cdot a^n$$

$$= (-1)^n \cdot n! \cdot a^n \cdot (ax+b)^{-(n+1)}$$

So $$\boxed{y_n = \frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}}$$

④ <u>nth derivative of $\ln(ax+b)$</u>

Let $y = \ln(ax+b)$

Diff. w.r.t. x successively

$$y_1 = \frac{1}{ax+b} \cdot a = (ax+b)^{-1} \cdot a$$

$$y_2 = (-1)(ax+b)^{-2} \cdot a^2$$

$$y_3 = (-1)(-2)(ax+b)^{-3} \cdot a^3$$

$$y_n = (-1)(-2)\cdots\cdots(-(n-1))(ax+b)^{-n} \cdot a^n$$

$$= (-1)^{n-1} \cdot (n-1)! \cdot (ax+b)^{-n} \cdot a^n$$

So $$\boxed{y_n = \frac{(-1)^{n-1} \cdot (n-1)! \cdot a^n}{(ax+b)^n}}$$

<u>Note</u> ① $\sin(\pi/2 + x) = \cos x$

② $\cos(\pi/2 + x) = -\sin x$

(5) nth derivative of $\sin(ax+b)$:- 50

Let $y = \sin(ax+b)$

Diff. w.r.t. x

$$y_1 = \cos(ax+b).a = a\cos(ax+b) = a\sin(ax+b+\pi/2)$$

$$y_2 = a\cos(ax+b+\pi/2)a = a^2\cos(ax+b+\pi/2) = a^2\sin(ax+b+2\pi/2)$$

$$y_3 = a^2\cos(ax+b+2\pi/2).a = a^3\cos(ax+b+2\tfrac{\pi}{2}) = a^3\sin(ax+b+3\pi/2)$$

$$\boxed{y_n = a^n\sin(ax+b+n\pi/2)}$$

(6) nth derivative of $\cos(ax+b)$:-

Let $y = \cos(ax+b)$

Diff. w.r.t. x

$$y_1 = -\sin(ax+b).a = a.-\sin(ax+b) = a\cos(ax+b+\pi/2)$$

$$y_2 = a.-\sin(ax+b+\pi/2).a = a^2.-\sin(ax+b+\pi/2) = a^2\cos(ax+b+2\pi/2)$$

$$y_3 = a^2.-\sin(ax+b+2\pi/2).a = a^3.-\sin(ax+b+2\pi/2) = a^3\cos(ax+b+3\tfrac{\pi}{2})$$

$$\boxed{y_n = a^n\cos(ax+b+n\pi/2)}$$

(7) nth derivative of $e^{ax}.\sin(bx+c)$:-

Let $y = e^{ax}.\sin(bx+c)$

Diff. w.r.t. x

$$y_1 = e^{ax}.\cos(bx+c).b + \sin(bx+c).e^{ax}.a$$

$$= e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$$

Put $a = r\cos\theta$ ——— (1)

$b = r\sin\theta$ ——— (2)

Sq. (1) & (2) & adding

$$a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta) \Rightarrow r^2 = a^2+b^2 \text{ or } r=\sqrt{a^2+b^2}$$

Dividing (2) by (1) $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

So above eq. becomes

$$y_1 = e^{ax}\left[r\cos\theta\cdot\sin(bx+c) + r\sin\theta\cdot\cos(bx+c)\right]$$

$$= re^{ax}\left[\sin(bx+c).\cos\theta + \cos(bx+c)\cdot\sin\theta\right]$$

$$y_1 = re^{ax}\cdot\sin(bx+c+\theta)$$

Again diff. w.r.t. x

$$y_2 = r\left[e^{ax}.\cos(bx+c+\theta)\cdot b + \sin(bx+c+\theta)\cdot e^{ax}\cdot a\right]$$

$$= re^{ax}\left[a\sin(bx+c+\theta) + b\cos(bx+c+\theta)\right]$$

$$= re^{ax}\left[r\cos\theta\cdot\sin(bx+c+\theta) + r\sin\theta\cdot\cos(bx+c+\theta)\right]$$

$$= r^2e^{ax}\left[\sin(bx+c+\theta).\cos\theta + \cos(bx+c+\theta)\cdot\sin\theta\right]$$

$$y_2 = r^2e^{ax}\sin(bx+c+2\theta)$$

Diff. w.r.t. x

$$y_3 = r^2\left[e^{ax}.\cos(bx+c+2\theta)\cdot b + \sin(bx+c+2\theta)\cdot e^{ax}\cdot a\right]$$

$$= r^2e^{ax}\left[a\sin(bx+c+2\theta) + b\cos(bx+c+2\theta)\right]$$

$$= r^2e^{ax}\left[r\cos\theta\cdot\sin(bx+c+2\theta) + r\sin\theta\cdot\cos(bx+c+2\theta)\right]$$

$$= r^3e^{ax}\left[\sin(bx+c+2\theta)\cdot\cos\theta + \cos(bx+c+2\theta)\cdot\sin\theta\right]$$

$$y_3 = r^3e^{ax}\cdot\sin(bx+c+3\theta)$$

$$y_n = r^ne^{ax}\cdot\sin(bx+c+n\theta)$$

or $y_n = \left[(a^2+b^2)^{1/2}\right]^n\cdot e^{ax}\cdot\sin(bx+c+n\theta)$

or $\boxed{y_n = (a^2+b^2)^{\frac{n}{2}}\cdot e^{ax}\cdot\sin(bx+c+n\tan^{-1}b/a)}$

$$y_3 = r^3 e^{ax}\left[\cos(bx+c+2\theta).\cos\theta - \sin(bx+c+2\theta).\sin\theta\right]$$ 53

$$= r^3 e^{ax}.\cos(bx+c+2\theta+\theta)$$

$$y_3 = r^3 e^{ax}\cos(bx+c+3\theta)$$

$$y_n = r^n e^{ax}.\cos(bx+c+n\theta)$$

$$=\left[(a^2+b^2)^{\frac{1}{2}}\right]^n \cdot e^{ax}.\cos(bx+c+n\theta)$$

So $$\boxed{y_n = (a^2+b^2)^{\frac{n}{2}}\cdot e^{ax}.\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)}$$

Leibniz's theorem:-

Statement:- If U & V are functions of x whose derivatives upto order n exist, then the nth derivative of their product is

$$[UV]^{(n)} = {}^{n}C_0\, U^{(n)}V + {}^{n}C_1\, U^{(n-1)}V' + {}^{n}C_2\, U^{(n-2)}.V'' + \cdots\cdots + {}^{n}C_n\, UV^{(n)}$$

Proof:- We will prove this theorem by applying principle of mathematical induction.

Step ① Put $n = 1$

$$L.H.S. = [UV]'$$
$$= U'V + UV'$$

& $$R.H.S. = {}^{1}C_0\, U'V + {}^{1}C_1\, UV'$$
$$= U'V + UV'$$

So $L.H.S. = R.H.S.$

Hence theorem is true for $n = 1$

So. C-1 is satisfied.

Step ② Suppose theorem is true for $n = r$

i.e.,

$$[UV]^{(r)} = {}^rC_0\, U^{(r)}V + {}^rC_1\, U^{(r-1)}V' + {}^rC_2\, U^{(r-2)}V'' + \cdots\cdots + {}^rC_r\, UV^{(r)}$$

Step③ Now we prove theorem for $n = r+1$

Diff. above eq. w.r.t. x

$$[UV]^{(r+1)} = {}^rC_0\left[U^{(r+1)}V + U^{(r)}V'\right] + {}^rC_1\left[U^{(r)}V' + U^{(r-1)}V''\right] + {}^rC_2\left[U^{(r-1)}V'' + U^{(r-2)}V'''\right] + \cdots\cdots + {}^rC_r\left[U'V^{(r)} + UV^{(r+1)}\right]$$

$$= {}^rC_0\, U^{(r+1)}V + {}^rC_0\, U^{(r)}V' + {}^rC_1\, U^{(r)}V' + {}^rC_1\, U^{(r-1)}V'' + {}^rC_2\, U^{(r-1)}V'' + {}^rC_2\, U^{(r-2)}V''' + \cdots\cdots + {}^rC_r\, U'V^{(r)} + {}^rC_r\, UV^{(r+1)}$$

$$= {}^rC_0\, U^{(r+1)}V + \left({}^rC_0 + {}^rC_1\right)U^{(r)}V' + \left({}^rC_1 + {}^rC_2\right)U^{(r-1)}V'' + \left({}^rC_2 + {}^rC_3\right)U^{(r-2)}V''' + \cdots\cdots + {}^rC_r\, UV^{(r+1)}$$

But ${}^nC_0 = {}^nC_n = 1$

& ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

So above eq. becomes

$$[UV]^{r+1} = {}^{r+1}C_0\, U^{(r+1)}V + {}^{r+1}C_1\, U^{(r)}V' + {}^{r+1}C_2\, U^{(r-1)}V'' + {}^{r+1}C_3\, U^{(r-2)}V''' + \cdots\cdots + {}^{r+1}C_{r+1}\, UV^{(r+1)}$$

Hence the theorem is true for $n = r+1$

So C-2 is satisfied.

Hence by principle of mathematical induction, the theorem is true for all +ve integers n.

Note By Leibniz's theorem

$$[UV]^{(n)} = {}^nC_0\, U^{(n)}V + {}^nC_1\, U^{(n-1)}V' + {}^nC_2\, U^{(n-2)}V'' + \cdots\cdots + {}^nC_n\, UV^{(n)}$$

As ${}^nC_0 = {}^nC_n = 1$ & ${}^nC_1 = {}^nC_{n-1} = n$, ${}^nC_2 = {}^nC_{n-2} = \dfrac{n(n-1)}{2!}$

So above eq. becomes

$$[UV]^{(n)} = U^{(n)}V + nU^{(n-1)}V' + \frac{n(n-1)}{2!}U^{(n-2)}V'' + \cdots\cdots + UV^{(n)}.$$

EXERCISE 2.5 (NEW BOOK)

EXERCISE 2.4 (OLD BOOK)

In **Problems** 1 – 4, find the nth order derivative :

1. $\dfrac{x}{x^2-a^2}$

Sol.

Let $y = \dfrac{x}{x^2-a^2}$

$y = \dfrac{x}{(x+a)(x-a)}$ —— ①

we resolve it into partial fraction

Now

$\dfrac{x}{(x+a)(x-a)} = \dfrac{A}{x+a} + \dfrac{B}{x-a}$

Multiplying both sides by $(x+a)(x-a)$

$x = A(x-a) + B(x+a)$ —— ②

To find A put $x = -a$ in ②

$-a = A(-a-a)$

$-a = -2aA$

$\boxed{A = \frac{1}{2}}$

To find B put $x = a$ in ②

$a = B(a+a)$

$a = 2aB$

$\boxed{B = \frac{1}{2}}$

So

$\dfrac{x}{(x+a)(x-a)} = \dfrac{\frac{1}{2}}{x+a} + \dfrac{\frac{1}{2}}{x-a}$

$= \dfrac{1}{2(x+a)} + \dfrac{1}{2(x-a)}$

Put in ①

$y = \dfrac{1}{2(x+a)} + \dfrac{1}{2(x-a)}$

Diff. w.r.t. x n times

$y^{(n)} = \dfrac{1}{2}\left[\dfrac{d^n}{dx^n}\left(\dfrac{1}{x+a}\right)\right] + \dfrac{1}{2}\left[\dfrac{d^n}{dx^n}\left(\dfrac{1}{x-a}\right)\right]$

$\because \dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n . n! . a^n}{(ax+b)^{n+1}}$

So $y^{(n)} = \dfrac{1}{2}\left[\dfrac{(-1)^n . n! . 1^n}{(x+a)^{n+1}}\right] + \dfrac{1}{2}\left[\dfrac{(-1)^n . n! . 1^n}{(x-a)^{n+1}}\right]$

$= \dfrac{(-1)^n . n!}{2}\left[\dfrac{1}{(x+a)^{n+1}} + \dfrac{1}{(x-a)^{n+1}}\right]$

2. $\dfrac{x^4}{(x-1)(x-2)}$

Sol.

Let $y = \dfrac{x^4}{(x-1)(x-2)}$

or $y = x^2+3x+7 + \dfrac{15x-14}{(x-1)(x-2)}$ —— ①

Now

$\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$

Multiplying both sides by $(x-1)(x-2)$

$15x-14 = A(x-2) + B(x-1)$ —— ②

To find A put $x = 1$ in ②

$15-14 = A(1-2)$

$1 = -A$

$\Rightarrow \boxed{A = -1}$

To find B put $x = 2$ in ②

$15(2)-14 = B(2-1)$

$30-14 = B$

$\boxed{B = 16}$

$$
\begin{array}{r|l}
x^2-3x+2 & x^4 \\
 & \underline{\cancel{x^4} - 3x^3 + 2x^2} \\
 & 3x^3 - 2x^2 \\
 & \underline{\cancel{3x^3} - 9x^2 + 6x} \\
 & 7x^2 - 6x \\
 & \underline{\cancel{7x^2} - 21x + 14} \\
 & 15x - 14
\end{array}
\qquad \text{Quotient: } x^2+3x+7
$$

So

$\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{16}{x-2}$

Put in ①

$y = (x^2+3x+7) + \dfrac{16}{x-2} - \dfrac{1}{x-1}$

Diff. w.r.t. x. n times

$y^{(n)} = 16 . \dfrac{d^n}{dx^n}\left(\dfrac{1}{x-2}\right) - \dfrac{d^n}{dx^n}\left(\dfrac{1}{x-1}\right)$

$$\because \frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n . n! . a^n}{(ax+b)^{n+1}}$$

So $$y^{(n)} = 16 . \frac{(-1)^n . n! . 1^n}{(x-2)^{n+1}} - \frac{(-1)^n . n! . 1^n}{(x-1)^{n+1}}$$

or $$y^{(n)} = (-1)^n . n! \left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$ —— Ans.

3. $y = e^{ax}\sin(bx + c)$

Sol. It has already been solved.

4. $e^{ax}\cos^2 x \sin x$

Sol:-

Let $y = e^{ax}\cos^2 x \sin x$

$= \frac{1}{2}\left[e^{ax}.(2\cos^2 x).\sin x\right]$

$= \frac{1}{2}\left[e^{ax}(1+\cos 2x).\sin x\right]$

$= \frac{1}{2}\left[e^{ax}(\sin x + \cos 2x . \sin x)\right]$

$= \frac{1}{2}\left[e^{ax}\sin x + e^{ax}.\cos 2x \sin x\right]$

$= \frac{1}{2}e^{ax}.\sin x + \frac{1}{2}\left(e^{ax}.\cos 2x \sin x\right)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\left(2\cos 2x \sin x\right)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\left(\sin(2x+x) - \sin(2x-x)\right)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\left(\sin 3x - \sin x\right)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x - \frac{1}{4}e^{ax}\sin x$

$= \left(\frac{1}{2} - \frac{1}{4}\right)e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$= \left(\frac{2-1}{4}\right)e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$= \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$y = \frac{1}{4}\left[e^{ax}\sin x + e^{ax}\sin 3x\right]$

Diff. w.r.t. x n times

$$y^{(n)} = \frac{1}{4}\left[\frac{d^n}{dx^n}\left(e^{ax}\sin x\right) + \frac{d^n}{dx^n}\left(e^{ax}\sin 3x\right)\right]$$

$\because \frac{d^n}{dx^n}\left(e^{ax}\sin(bx+c)\right) = (a^2+b^2)^{\frac{n}{2}}.e^{ax}\sin(bx+c+n\tan^{-1}\frac{b}{a})$

So

$$y^{(n)} = \frac{1}{4}\left[(a^2+1)^{\frac{n}{2}}.e^{ax}\sin(x+n\tan^{-1}\frac{1}{a}) + (a^2+3^2)^{\frac{n}{2}}.e^{ax}\sin(3x+n\tan^{-1}\frac{3}{a})\right]$$

or

$$y^{(n)} = \frac{1}{4}\left[(a^2+1)^{\frac{n}{2}}.e^{ax}\sin(x+n\tan^{-1}\frac{1}{a}) + (a^2+9)^{\frac{n}{2}}e^{ax}\sin(3x+n\tan^{-1}\frac{3}{a})\right]$$

5. If $y = \arc\tan x$, show that

$$(1+x^2)y^{(2)} + 2xy' = 0$$

Hence find the values of all derivatives of y when $x = 0$

Sol.

Let $y = \tan^{-1}x$

Diff. w.r.t. x

$$y' = \frac{1}{1+x^2} \quad \text{——— (1)}$$

or $(1+x^2)y' = 1$

w.r.t. x

$$(1+x^2)y'' + 2xy' = 0 \quad \text{——— (2)}$$

Diff. w.r.t. x n times

$$[y''\cdot(1+x^2)]^{(n)} + [2y'x]^{(n)} = 0$$

Using Leibniz's theorem

$$(y'')^{(n)}(1+x^2) + n(y'')^{(n-1)}\cdot 2x + \frac{n(n-1)}{2!}(y'')^{(n-2)}\cdot 2 + 2\left[(y')^{(n)}\cdot x + n(y')^{(n-1)}\cdot 1\right] = 0$$

$$(1+x^2)y^{(n+2)} + 2nxy^{(n+1)} + (n^2-n)y^{(n)} + 2xy^{(n+1)} + 2ny^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+2)xy^{(n+1)} + (n^2-n+2n)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+2)xy^{(n+1)} + (n^2+n)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+2)xy^{(n+1)} + n(n+1)y^{(n)} = 0 \quad \text{——— (3)}$$

Put $x=0$ in (1), (2) + (3)

$$\left.\begin{aligned} y'(0) &= 1 \\ y''(0) &= 0 \\ y^{(n+2)}(0) &= -n(n+1)y^{(n)}(0) \end{aligned}\right\}$$

58

For $n=1$, $y^{(3)}(0) = -1\cdot 2\, y'(0) = -2\cdot 1 \Rightarrow y^{2(1)+1}(0) = (-1)^1\cdot 2!$

For $n=2$, $y^{(4)}(0) = -2\cdot 3\, y''(0) = -2\cdot 3\cdot 0 = 0 \Rightarrow y^{2(2)}(0) = 0$

For $n=3$, $y^{(5)}(0) = -3\cdot 4\cdot y^{(3)}(0) = -3\cdot 4\cdot -2\cdot 1 \Rightarrow y^{2(2)+1}(0) = (-1)^2\cdot 4!$

For $n=4$, $y^{(6)}(0) = -4\cdot 5\cdot y^{(4)}(0) = -4\cdot 5\cdot 0 = 0 \Rightarrow y^{2(3)}(0) = 0$

For $n=5$, $y^{(7)}(0) = -5\cdot 6\cdot y^{(5)}(0) = -5\cdot 6\cdot (-1)^2\cdot 4! \Rightarrow y^{2(3)+1}(0) = (-1)^3\cdot 6!$

For $n=6$, $y^{(8)}(0) = -6\cdot 7\cdot y^{(6)}(0) = -6\cdot 7\cdot 0 = 0 \Rightarrow y^{2(4)}(0) = 0$

On generalizing we get

$$y^{2(n)+1}(0) = (-1)^n\cdot(2n)!$$

$$y^{2(n)}(0) = 0$$

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6. If $y = \sin(a \text{ arc} \sin x)$, prove that

$$(1-x^2)\, y^{(n+2)} = (2n+1)\, xy^{(n+1)} - (n^2 - a^2)\, y^{(n)}$$

Sol. $y = \sin(\sin^{-1} x)$

Diff. w.r.t. x

$$y' = \cos(\sin^{-1} x) \cdot \frac{a}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y' = a\cos(\sin^{-1} x)$$

$$(1-x^2) y'^2 = a^2 \cos^2(\sin^{-1} x)$$

$$(1-x^2) y'^2 = a^2(1 - \sin^2(\sin^{-1} x))$$

or $(1-x^2) y'^2 = a^2(1-y^2)$

Diff. w.r.t. x

$$(1-x^2) \cdot 2y'y'' + (-2x) y'^2 = -2yy'a^2$$

Dividing both sides by $2y'$

$$(1-x^2)y'' - xy' = -a^2 y$$

Diff. w.r.t. x n times

$$[y''(1-x^2)]^{(n)} - [y'x]^{(n)} = -a^2 y^{(n)}$$

using Leibniz's theorem

$$(y'')^{(n)}(1-x^2) + n(y'')^{(n-1)}(-2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}(-2) - \left[(y')^{n} \cdot x + n(y')^{(n-1)} \cdot 1\right] = -a^2 y^{(n)}$$

$$(1-x^2) y^{(n+2)} - 2nx\, y^{(n+1)} - (n^2-n)\, y^{(n)} - x y^{(n+1)} - n y^{(n)} + a^2 y^{(n)} = 0$$

$$(1-x^2) y^{(n+2)} - (2n+1) x\, y^{(n+1)} - (n^2 - n + n - a^2)\, y^{(n)} = 0$$

$$(1-x^2) y^{(n+2)} - (2n+1) x\, y^{(n+1)} - (n^2 - a^2)\, y^{(n)} = 0$$

7. If $y = e^{m \text{ arc} \sin x}$, show that

$$(1-x^2)\, y^{(n+2)} - (2n+1)\, xy^{(n+1)} - (n^2 + m^2)\, y^{(n)} = 0.$$

Find the value of y^n at $x = 0$

Sol. $y = e^{m\sin^{-1} x}$

Diff. w.r.t. x

$$y' = e^{m\sin^{-1}x} \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{———} (1)$$ 6.

$$\sqrt{1-x^2}\, y' = m e^{m\sin^{-1}x}$$

$$\sqrt{1-x^2}\, y' = my$$

Sq. both sides

$$(1-x^2)y'^2 = m^2y^2$$

Diff. w.r.t. x

$$(1-x^2).2y'y'' + (-2x)y'^2 = m^2(2yy')$$

Dividing both sides by $2y'$

$$(1-x^2)y'' - xy' = m^2 y \quad \text{———} (2)$$

Diff. w.r.t. x n times

$$\left[y''(1-x^2)\right]^{(n)} - \left[y'x\right]^{(n)} = m^2 y^{(n)}$$

using Leibniz's theorem

$$(y'')^{(n)}(1-x^2) + n(y'')^{(n-1)}(-2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}(-2) - \left[(y')^{(n)}\cdot x + n(y')^{(n-1)}\cdot 1\right] = m^2 y^{(n)}$$

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - (n^2-n)y^{(n)} - xy^{(n+1)} - ny^{(n)} - m^2y^{(n)} = 0$$

$$(1-x^2)y^{(n+1)} - (2n+1)xy^{(n+1)} - (n^2-n+n+m^2)y^{(n)} = 0$$

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2+m^2)y^{(n)} = 0 \quad \text{———} (3)$$

Put $x=0$ in (1), (2) & (3)

$$\left.\begin{aligned} y'(0) &= m \\ y''(0) &= m^2 \\ y^{(n+2)}(0) &= (n^2+m^2)\,y^{(n)}(0) \end{aligned}\right\}$$

For $n=1$, $y^{(3)}(0) = (1^2+m^2)y'(0) = (1^2+m^2).m$

For $n=2$, $y^{(4)}(0) = (2^2+m^2)y''(0) = (2^2+m^2)m^2$

For $n=3$, $y^{(5)}(0) = (3^2+m^2)y^{(3)}(0) = (3^2+m^2)(1^2+m^2).m$

For $n=4$, $y^{(6)}(0) = (4^2+m^2)y^{(4)}(0) = (4^2+m^2)(2^2+m^2).m^2$

For $n=5$, $y^{(7)}(0) = (5^2+m^2)y^{(5)}(0) = (5^2+m^2)(3^2+m^2)(1^2+m^2).m$

For $n=6$, $y^{(8)}(0) = (6^2+m^2)y^{(6)}(0) = (6^2+m^2)(4^2+m^2)(2^2+m^2).m^2$

on generalizing we have

$y_n(0) = [(n-2)^2+m^2]\text{--------}(4^2+m^2)(2^2+m^2).m^2$ if n is even

$y_n(0) = [(n-2)^2+m^2]\text{--------}(3^2+m^2)(1^2+m^2).m$ if n is odd

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8. Find $y^{(n)}(0)$ if

(a) $y = \ln[x+\sqrt{1+x^2}]$

(b) $y = \ln(x+\sqrt{1+x^2})^m$

Sol. **(a)** $y = \ln(x+\sqrt{1+x^2})$

Diff. w.r.t. x

$$y' = \frac{1}{(x+\sqrt{1+x^2})}.\left(1+\frac{1}{2\sqrt{1+x^2}}.2x\right)$$

$$y' = \frac{1}{(x+\sqrt{1+x^2})}.\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

62

$$y' = \frac{1}{(\cancel{x+\sqrt{1+x^2}})}\cdot\left(\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\right)$$

$$y' = \frac{1}{\sqrt{1+x^2}} \quad \text{———} \; (1)$$

$$\sqrt{1+x^2}\, y' = 1$$

$$(1+x^2)y'^2 = 1$$

Diff. w.r.t. x

$$(1+x^2).2y'y'' + 2xy'^2 = 0$$

$$(1+x^2)y'' + xy' = 0$$

Diff. w.r.t. x n times

$$\left[y''(1+x^2)\right]^{(n)} + \left[y'x\right]^{(n)} = 0$$

By Leibniz's theorem

$$(y'')^{(n)}(1+x^2) + n(y'')^{(n-1)}.(2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}.(2) + (y')^{(n)}.x + n(y')^{(n-1)}.1 = 0$$

$$(1+x^2).y^{(n+1)} + 2nx\,y^{(n+1)} + (n^2-n)\,y^{(n)} + x\,y^{(n+1)} + n\,y^{(n)} = 0$$

$$(1+x^2)\,y^{(n+2)} + (2n+1)x\,y^{(n+1)} + (n^2-n+n)\,y^{(n)} = 0$$

$$(1+x^2)\,y^{(n+2)} + (2n+1)x\,y^{(n+1)} + n^2 y^{(n)} = 0 \quad \text{———} \; (3)$$

Put $x=0$ in (1), (2) & (3)

$$\left.\begin{aligned} y'(0) &= 1 \\ y''(0) &= 0 \\ y^{(n+2)}(0) &= -n^2 y^{(n)}(0) \end{aligned}\right\}$$

For $n=1$, $y^{(3)}(0) = -(1)^2 y'(0) = -(1)^2.1 \Rightarrow y^{2(1)+1}(0) = (-1)^1.1^2$

For $n=2$, $y^{(4)}(0) = -(2)^2.y''(0) = -(2)^2.0 = 0 \Rightarrow y^{2(2)}(0) = 0$

For $n=3$, $y^{(5)}(0) = -(3)^2.y^{(3)}(0) = -(3)^2.-(1)^2 \Rightarrow y^{2(2)+1}(0) = (-1)^2.1^2.3^2$

For $n=4$, $y^{(6)}(0) = -(4)^2.y^{(4)}(0) = -(4)^2.0 = 0 \Rightarrow y^{2(3)}(0) = 0$

For $n=5$, $y^{(7)}(0) = -(5)^2.y^{(5)}(0) = -(5)^2.-(3)^2.-(1)^2 \Rightarrow y^{2(3)+1}(0) = (-1)^3.1^2.3^2.5^2$

For $n=6$, $y^{(8)}(0) = -(6)^2.y^{(6)}(0) = -(6)^2.0 = 0 \Rightarrow y^{2(4)}(0) = 0$

on generalizing we get

$$y^{(2n+1)}(0) = (-1)^n.\,1^2.3^2.5^2.\,\ldots\ldots(2n-1)^2$$

$$y^{(2n)}(0) = 0$$

Sol. (h) $y = (x+\sqrt{1+x^2})^m$

Diff. w.r.t. x

$$y' = m(x+\sqrt{1+x^2})^{m-1}\cdot\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot(x+\sqrt{1+x^2})\cdot\frac{1}{\sqrt{1+x^2}}$$

$$y' = m(x+\sqrt{1+x^2})^{m}\cdot\frac{1}{\sqrt{1+x^2}}$$

$$y' = \frac{my}{\sqrt{1+x^2}} \quad \text{——— (1)}$$

$$\sqrt{1+x^2}\,y' = my$$

Sq. both sides

$$(1+x^2)y'^2 = m^2y^2$$

Diff. w.r.t. x

$$(1+x^2).2y'y'' + 2xy'^2 = m^2(2yy')$$

Dividing both sides by $2y'$

$$(1+x^2)y'' + xy' = m^2y \quad \text{——— (2)}$$

Diff. w.r.t. x n times

$$\left[y''(1+x^2)\right]^{(n)} + \left[y'x\right]^{(n)} = m^2y^{(n)}$$

using Leibniz's theorem

$$(y'')^{(n)}(1+x^2) + n(y'')^{(n-1)}\cdot(2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}\cdot 2 + (y')^{(n)}\cdot x + n(y')^{(n-1)}\cdot 1 = m^2y^{(n)}$$

$$(1+x^2)y^{(n+2)} + 2nx\,y^{(n+1)} + (n^2-n)y^{(n)} + xy^{(n+1)} + ny^{(n)} - m^2y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+1)x\,y^{(n+1)} + (n^2-n+n-m^2)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0$$

Put $x=0$ in ①, ②, ③

$$\left.\begin{aligned} y'(0) &= m \\ y''(0) &= m^2 \\ y^{(n+2)}(0) &= (m^2-n^2)\,y^{(n)}(0) \end{aligned}\right\}$$

For $n=1$, $y^{(3)}(0) = (m^2-1^2)\,y'(0) = (m^2-1^2).m \Rightarrow y^{2(1)+1}(0) = (m^2-1^2).m$

For $n=2$, $y^{(4)}(0) = (m^2-2^2)\,y''(0) = (m^2-2^2).m^2 \Rightarrow y^{2(2)}(0) = (m^2-2^2).m^2$

For $n=3$, $y^{(5)}(0) = (m^2-3^2).y^{(3)}(0) = (m^2-3^2)(m^2-1^2).m \Rightarrow y^{2(2)+1}(0) = (m^2-3^2)(m^2-1^2).m$

For $n=4$, $y^{(6)}(0) = (m^2-4^2).y^{(4)}(0) = (m^2-4^2)(m^2-2^2)m^2 \Rightarrow y^{2(3)} = (m^2-4^2)(m^2-2^2).m^2$

On generalizing we have

$$y^{2n+1}(0) = \left(m^2-(2n-1)^2\right)\text{---------}(m^2-3^2)(m^2-1^2).m$$

$$y^{2n}(0) = \left(m^2-(2n-2)^2\right)\text{---------}(m^2-4^2)(m^2-2^2).m^2$$

<u>Q9</u> If $f(x) = \ln(1+\sqrt{1-x})$, Prove that

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

Sol:-

$$f(x) = \ln(1+\sqrt{1-x})$$

Diff. w.r.t. x

$$f'(x) = \frac{1}{(1+\sqrt{1-x})}\cdot\frac{1}{2\sqrt{1-x}}(-1)$$

or $$f'(x) = \frac{1}{(1+\sqrt{1-x})}\cdot\frac{-1}{2\sqrt{1-x}}$$

or $$2\sqrt{1-x}\,f'(x) = \frac{-1}{(1+\sqrt{1-x})}\times\frac{(1-\sqrt{1-x})}{(1-\sqrt{1-x})}$$

$$= \frac{-(1-\sqrt{1-x})}{1-(1-x)}$$

$$= \frac{-(1-\sqrt{1-x})}{1-1+x}$$

$$2\sqrt{1-x}\, f'(x) = \frac{-1+\sqrt{1-x}}{x}$$

$$2x\sqrt{1-x}\, f'(x) = -1+\sqrt{1-x}$$

Diff. w.r.t. x

$$2\left[x\sqrt{1-x}\, f''(x) + f'(x)\left(x\cdot\frac{1}{2\sqrt{1-x}}(-1) + \sqrt{1-x}\cdot 1\right)\right] = \frac{1}{2\sqrt{1-x}}(-1)$$

$$2x\sqrt{1-x}\, f''(x) + 2f'(x)\left(\frac{-x}{2\sqrt{1-x}} + \sqrt{1-x}\right) = \frac{-1}{2\sqrt{1-x}}$$

Multiplying both sides by $2\sqrt{1-x}$

$$4x(1-x)f''(x) + 4f'(x)\sqrt{1-x}\left(\frac{-x+2(1-x)}{2\sqrt{1-x}}\right) = -1$$

$$4x(1-x)f''(x) + 2f'(x)(-x+2-2x)+1 = 0$$

$$4x(1-x)f''(x) + 2f'(x)(-3x+2)+1 = 0$$

or $4x(1-x)f''(x) + 2(2-3x)f'(x)+1 = 0$

---

10. If $y = a\cos(\ln x)+b\sin(\ln x)$, prove that

$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$$

Sol. $y = a\cos(\ln x)+b\sin(\ln x)$

Diff. w.r.t. x

$$y' = -a\sin(\ln x)\cdot\frac{1}{x} + b\cos(\ln x)\cdot\frac{1}{x}$$

$$xy' = -a\sin(\ln x) + b\cos(\ln x)$$

Diff. w.r.t. x

$$xy'' + y'\cdot 1 = -a\cos(\ln x)\cdot\frac{1}{x} - b\sin(\ln x)\cdot\frac{1}{x}$$

$$x^2y'' + xy' = -\left(a\cos(\ln x)+b\sin(\ln x)\right)$$

$$x^2y'' + xy' = -y$$

$$x^2y'' + xy' + y = 0$$

Diff. w.r.t. x n times

$$\left[y''\cdot x^2\right]^{(n)} + \left[y'\cdot x\right]^{(n)} + y^{(n)} = 0$$

Using Leibniz's theorem

$$(y'')^{(n)}.x^2+n(y'')^{(n-1)}.2x+\frac{n(n-1)}{2!}(y'')^{(n-2)}.2+(y')^{(n)}.x+n(y')^{(n-1)}.1+y^{(n)}=0$$

$$x^2y^{(n+2)}+2nxy^{(n+1)}+(n^2-n)y^{(n)}+xy^{(n+1)}+ny^{(n)}+y^{(n)}=0$$

$$x^2y^{(n+2)}+(2n+1)xy^{(n+1)}+(n^2-n+n+1)y^{(n)}=0$$

$$x^2y^{(n+2)}+(2n+1)xy^{(n+1)}+(n^2+1)y^{(n)}=0$$

11. If $x^y = e^{x-y}$, find $\frac{d^n y}{dx^n}$.

Sol. $x^y = e^{x-y}$

taking ln on both sides

$$\ln x^y = \ln e^{x-y}$$

$$y\ln x = (x-y)\ln e$$

$$y\ln x = x-y$$

$$y+y\ln x = x$$

$$y(1+\ln x) = x$$

Diff. w.r.t. x n times

$$[y(1+\ln x)]^{(n)} = 0$$

$$y^{(n)}(1+\ln x)+ny^{(n-1)}.\frac{1}{x}+\frac{n(n-1)}{2!}.y^{(n-2)}.\left(\frac{-1}{x^2}\right)+------+y\frac{(-1)^{n-1}\cdot(n-1)!\cdot 1^n}{x^n}=0$$

$$y^{(n)}(1+\ln x)+\frac{n}{x}y^{(n-1)}-\frac{n(n-1)}{2x^2}y^{(n-2)}+-------+y.\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}=0$$

12. Show that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right)=\frac{(-1)^n n!}{x^{n+1}}\left[\ln x-1-\frac{1}{2}-\frac{1}{3}\cdots-\frac{1}{n}\right]$$

Sol:-

Let $U=\frac{1}{x}$ & $V=\ln x$

As we know that $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right)=\frac{(-1)^n.n!.a^n}{(ax+b)^{n+1}}$ & $\frac{d^n}{dx^n}\left(\ln(ax+b)\right)=\frac{(-1)^{n-1}.(n-1)!.a^n}{(ax+b)^n}$

So $U^{(n)}=\frac{(-1)^n.n!}{x^{n+1}}$ & $V^{(n)}=\frac{(-1)^{n-1}.(n-1)!}{x^n}$

By Leibniz's theorem, we have

$$[UV]^{(n)}=U^{(n)}.V+nU^{(n-1)}.V'+\frac{n(n-1)}{2!}U^{(n-2)}V''+.......+nU'V^{(n-1)}+UV^{(n)}$$

Putting values we get

$$\left[\frac{1}{x}.\ln x\right]^{(n)}=\frac{(-1)^n.n!}{x^{n+1}}.\ln x+n.\frac{(-1)^{n-1}.(n-1)!}{x^n}.\frac{1}{x}+\frac{n(n-1)}{2!}.\frac{(-1)^{n-2}.(n-2)!}{x^{n-1}}.\frac{-1}{x^2}$$
$$+\frac{n(n-1)(n-2)}{3!}.\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}.\frac{2}{x^3}+....+\frac{1}{x}.\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}$$

67

$$\left(\frac{\ln x}{x}\right)^{(n)} = \frac{(-1)^n \cdot n!}{x^{n+1}} \cdot \ln x + \frac{(-1)^{n-1} \cdot n(n-1)!}{x^{n+1}} - \frac{(-1)^{n-2} \cdot n(n-1)(n-2)!}{2! \cdot x^{n+1}} + \frac{(-1)^{n-3} 2n(n-1)(n-1)(n-3)!}{3! \cdot x^{n+1}} + \cdots + \frac{(-1)^{n-1} \cdot (n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n \cdot n! \cdot \ln x}{x^{n+1}} + \frac{(-1)^n \cdot (-1)^{-1} \cdot n!}{x^{n+1}} - \frac{(-1)^n \cdot (-1)^{-2} \cdot n!}{2x^{n+1}} + \frac{(-1)^n \cdot (-1)^{-3} \cdot n!}{3x^{n+1}} + \cdots + \frac{(-1)^n \cdot (-1)^{-1} \cdot n(n-1)!}{n x^{n+1}}$$

$$= \frac{(-1)^n \cdot n! \cdot \ln x}{x^{n+1}} - \frac{(-1)^n \cdot n!}{x^{n+1}} - \frac{(-1)^n \cdot n!}{2x^{n+1}} - \frac{(-1)^n \cdot n!}{3x^{n+1}} - \cdots - \frac{(-1)^n \cdot n!}{n x^{n+1}}$$

$$\left(\frac{\ln x}{x}\right)^{(n)} = \frac{(-1)^n \cdot n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right]$$

Techniques of Integration

EXERCISE 4.2

Q No. 1 $I = \int \frac{dx}{\sqrt{a^2+x^2}}$

Put $x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta d\theta$

$$I = \int \frac{a\sec^2\theta d\theta}{\sqrt{(a^2 + a^2\tan^2\theta)}}$$

$$= \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2(1+\tan^2\theta)}} = \int \frac{a\sec^2\theta d\theta}{a\sqrt{(1+\tan^2\theta)}} =$$

$$\int \frac{\sec^2\theta d\theta}{\sqrt{(1+\tan^2\theta)}}$$

$$= \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta|$$

Now substitution returns:

$$I = \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right|$$

$$I = \ln\left|\sqrt{1+\left(\frac{x}{a}\right)^2} + \frac{x}{a}\right| = \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|$$

Q No. 2 $I = \int \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\cosh\theta \quad \Rightarrow \quad dx = a\sinh\theta d\theta$

$$I = \int \frac{a\sinh\theta d\theta}{\sqrt{(a^2\cosh^2\theta - a^2)}}$$

$$= \int \frac{a\sinh\theta d\theta}{\sqrt{a^2(\cosh^2\theta - 1)}} = \int \frac{a\sinh\theta d\theta}{a\sqrt{(\cosh^2\theta - 1)}}$$

$$= \int \frac{\sinh\theta d\theta}{\sqrt{(\cosh^2\theta - 1)}} = \int \frac{\sinh\theta d\theta}{\sinh\theta}$$

$$= \int d\theta = \theta$$

Now substitution returns:

$$= \cosh^{-1}\frac{x}{a}$$

Q No. 3 $I = \int \tan x dx$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{-\sin x}{\cos x} dx = -\ln(\cos x) = \ln(\sec x)$$

Q No. 4 $I = \int \cot x dx$

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln(\sin x)$$

Q No. 5 $I = \int \sec x dx$

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{\sec^2 x + \sec x\tan x)}{(\sec x + \tan x)} dx = \ln(\sec x + \tan x)$$

Q No. 6 $I = \int \csc x dx$

$$\int \csc x dx = -\int \frac{-\csc x(\csc x + \cot x)}{(\csc x + \cot x)} dx$$

$$= -\int \frac{-\mathrm{cosec}^2 x - \csc x\cot x)}{(\csc x + \cot x)} = -\ln(\csc x + \cot x)$$

by rationalizing this answer we can get another result

i.e $\ln(\csc x - \cot x)$

Q No. 7 $I = \int (ax^2 + 2bx + c)^2(ax + b)dx$

$$I = \frac{1}{2}\int (ax^2 + 2bx + c)^2(2ax + 2b)dx$$

$$I = \frac{(ax^2 + 2bx + c)^{2+1}}{2+1}$$

Q No. 8 $I = \int \sqrt{\frac{1+x}{1-x}}\,dx$

By rationalizing we get, $\frac{1+x}{1-x} \times \frac{1+x}{1+x} = \frac{(1+x)^2}{1-x^2}$

So, $I = \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{xdx}{\sqrt{1-x^2}}$

$= sin^{-1}x + \int (1-x^2)^{-\frac{1}{2}}xdx$

$= sin^{-1}x - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x)dx$

$= sin^{-1}x - \frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$= sin^{-1}x - \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}$

$= sin^{-1}x - \sqrt{1-x^2}$

Q No. 9 $\int \frac{dx}{a+\sqrt{bx+c}}$

(linear under square root)

Put $\sqrt{bx+c} = z$

$\Rightarrow\ bx + c = z^2$

$\Rightarrow\ b\,dx = 2zdz$

$I = \int \frac{2zdz/b}{a+z}$

$I = \frac{2}{b}\int \frac{zdz}{a+z}$

$I = \frac{2}{b}\int \left(1 - \frac{a}{a+z}\right)dz$

$I = \frac{2}{b}\int dz - \frac{2a}{b}\int \frac{dz}{a+z}$

$I = \frac{2}{b}z - \frac{2a}{b}ln(a+z)$

$I = \frac{2}{b}\sqrt{bx+c} - \frac{2a}{b}ln\,(a+\sqrt{bx+c})$

Q No. 10 $\int \frac{dx}{(1+x^2)tan^{-1}x}$

$I = \int \frac{1/(1+x^2)}{tan^{-1}x}dx \quad = \ln(tan^{-1}x)$

Q No. 11 $I = \int \frac{sinx+cosx}{sinx-cosx}dx$

$I = \int \frac{cosx - (-sinx)}{sinx - cosx}dx \ = \ln(sinx - cosx)$

Q No. 12 $I = \int \frac{sin\sqrt{x}}{\sqrt{x}}dx$

(Substitute the complicated angle)

Put $\sqrt{x} = z \ \Rightarrow\ x = z^2 \ \Rightarrow\ dx = 2zdz$

$I = \int \frac{sinz}{z}.2zdz \ = 2\int sinzdz \ = -2cosz$

$= -2cos\sqrt{x}$

Q No. 13 $I = \int \sqrt{e^{2x}+e^{3x}}dx$

$I = \int \sqrt{e^{2x}+e^{3x}}dx = \int \sqrt{e^{2x}(1+e^x)}dx$

$I = \int \sqrt{1+e^x}.e^x dx \ = \frac{(1+e^x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

Q No. 14 $I = \int \frac{dx}{e^x+e^{-x}}$

I

$= \int \frac{e^x dx}{e^x(e^x+e^{-x})}$ $\quad$ *(multiplied D^r and N^r by e^x)*

$I = \int \frac{e^x dx}{e^{2x}+1} \ = \tan^{-1}(e^x)$

<u>Alternatively,</u>

Put $e^x = z \quad \Rightarrow e^x dx = dz$

So $I = \int \frac{e^x dx}{e^{2x}+1} = \int \frac{dz}{z^2+1} = \tan^{-1}z = \tan^{-1}(e^x)$

Q No. 15 $I = \int \frac{e^{2x}dx}{\sqrt{e^x-1}}$

$$Put\ e^x = z \quad \Rightarrow e^x dx = dz$$

$$I = \int \frac{e^x.e^x dx}{\sqrt{e^x-1}} = \int \frac{zdz}{\sqrt{z-1}} = \int \frac{(z-1+1)dz}{\sqrt{z-1}}$$

$$I = \int \frac{(z-1)dz}{\sqrt{z-1}} + \int \frac{dz}{\sqrt{z-1}}$$

$$I = \int (z-1)^{1-\frac{1}{2}}dz + \int (z-1)^{\frac{-1}{2}}dz$$

$$I = \int (z-1)^{\frac{1}{2}}dz + \int (z-1)^{\frac{-1}{2}}dz$$

$$I = \frac{(z-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(z-1)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}$$

$$I = \frac{(z-1)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(z-1)^{\frac{1}{2}}}{\frac{1}{2}} = \frac{2}{3}(e^x-1)^{\frac{3}{2}} + 2\sqrt{e^x-1}$$

Q No. 16 $I = \int \frac{\cos(\ln x)}{x}dx$

(Substitute the complicated angle)

Put $\ln x = z \quad \Rightarrow \quad \frac{1}{x}\,dx = dz$

$$I = \int cosz dz = sinz = \sin(\ln x)$$

Q No. 17 $I = \int \frac{2x+5}{\sqrt{x^2+5x+7}}dx$

$$I = \int (x^2+5x+7)^{-\frac{1}{2}}.(2x+5)dx$$

$$I = \frac{(x^2+5x+7)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

Q No. 18 $I = \int \frac{x+2}{\sqrt{2x^2+8x+5}}dx$

$$I = \int (2x^2+8x+5)^{-\frac{1}{2}}.(x+2)dx$$

$$I = \frac{1}{4}\int (2x^2+8x+5)^{-\frac{1}{2}}.4(x+2)dx$$

$$I = \frac{1}{4}\int (2x^2+8x+5)^{-\frac{1}{2}}.(4x+8)dx$$

$$I = \frac{(2x^2+8x+5)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

Q No. 19 $I = \frac{\sqrt{x^2-a^2}}{x^4}dx$

Put $x = acosh\theta \quad \Rightarrow \quad dx = asinh\theta$

$$I = \int \frac{\sqrt{a^2\cosh^2\theta - a^2}}{a^4\cosh^4\theta}asinh\theta d\theta$$

$$I = \int \frac{a\sqrt{\cosh^2\theta - 1}}{a^4\cosh^4\theta}asinh\theta d\theta$$

$$I = \int \frac{\sqrt{\cosh^2\theta - 1}}{a^2\cosh^4\theta}\sqrt{\cosh^2\theta - 1}\,d\theta$$

$$I = \frac{1}{a^2}\int \frac{\cosh^2\theta - 1}{\cosh^4\theta}d\theta$$

$$I = \frac{1}{a^2}\int \frac{1}{\cosh^2\theta}d\theta - \frac{1}{a^2}\int \frac{1}{\cosh^4\theta}d\theta$$

$$I = \frac{1}{a^2}\int sech^2\theta d\theta - \frac{1}{a^2}\int sech^4\theta d\theta$$

$$I = \frac{1}{a^2}tanh\theta - \frac{1}{a^2}I_1 \text{-------------(1)}$$

$$I_1 = \int sech^4\theta d\theta$$

$$I_1 = \int sech^2\theta.sech^2\theta d\theta$$

$$I_1 = \int (1-tanh^2\theta).sech^2\theta d\theta$$

$$I_1 = \int sech^2\theta d\theta - \int tanh^2\theta sech^2\theta d\theta$$

$$I_1 = tanh\theta - \frac{tanh^3\theta}{3}$$

Putting in eq(1) we get,

$$I = \frac{1}{a^2}tanh\theta - \frac{1}{a^2}tanh\theta + \frac{tanh^3\theta}{3a^2}$$

$$I = \frac{tanh^3\theta}{3} = \frac{1}{3}.\left(\frac{sinh\theta}{cosh\theta}\right)^3 = \frac{1}{3}.\left(\frac{\sqrt{cosh^2\theta - 1}}{cosh\theta}\right)^3$$

$$I = \frac{1}{3a^2}\left(\frac{\sqrt{\frac{x^2}{a^2} - 1}}{\frac{x}{a}}\right)^3 = \frac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3}$$

Q No. 20 $I = \int cos^6 x sin^3 x dx$

$$I = \int cos^6 x . sin^2 x . sinx\, dx$$

$$= \int cos^6 x . (1 - cos^2 x) . sinx\, dx$$

$$= \int cos^6 x \;\; . sinx\, dx - \int cos^8 x \;\; . sinx\, dx$$

$$= -\int cos^6 x \;\; . (-sinx) dx + \int cos^8 x \;\; . (-sinx)\, dx$$

$$= -\frac{cos^7 x}{7} + \frac{cos^9 x}{9}$$

Q No. 21 $I = \int tan^3 x sec^3 x dx$

$$I = \int tan^2 x . sec^2 x . secxtanx\, dx$$

$$I = \int (sec^2 x - 1) . sec^2 x . secxtanx\, dx$$

$$I = \int sec^4 x . secxtanx\, dx - \int sec^2 x . secxtanx\, dx$$

$$I = \frac{sec^5 x}{5} - \frac{sec^3 x}{3}$$

Q No. 22 $I = \int cot^3 x csc^4 x dx$

$$I = \int cot^2 x csc^3 x . (cotxcscx) dx$$

$$I = -\int cot^2 x csc^3 x . (-cotxcscx) dx$$

$$I = -\int (csc^2 x - 1) csc^3 x . (-cotxcscx) dx$$

$$I = -\int csc^5 x\, (-cotxcscx) dx + \int csc^3 x (-cotxcscx) dx$$

$$I = \frac{-csc^6 x}{6} + \frac{csc^4 x}{4}$$

Alternatively,

$$I = \int cot^3 x csc^4 x dx$$

$$I = \int cot^3 x csc^2 x . (csc^2 x) dx$$

$$I = -\int cot^3 x csc^2 x . (-csc^2 x) dx$$

$$I = -\int cot^3 x (cot^2 x + 1) . (-csc^2 x) dx$$

$$I = -\int cot^5 x\, (-csc^2 x) dx - \int cot^3 x\, (-csc^2 x) dx$$

$$I = -\frac{1}{6} cot^6 x - \frac{1}{4} cot^4 x$$

Q No. 23 $I = \int \frac{dx}{\sqrt{2x^2+3x+4}}$

$$I = \int \frac{dx}{\sqrt{2(x^2 + \frac{3}{2}x + 2)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x^2 + \frac{3}{2}x + 2)}}$$

Completing square of

$$x^2 + \frac{3}{2}x + 2$$

$$= (x)^2 + 2\left(\frac{3}{4}\right)(x) + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 + 2$$

$$= \left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + 2$$

$$= \left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{32}{16}$$

$$= \left(x + \frac{3}{4}\right)^2 + \frac{23}{16}$$

$$= \left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2$$

So,

$$I = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}}$$

$$I = \frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{x + \frac{3}{4}}{\frac{\sqrt{23}}{4}}\right) = \frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{4x + 3}{\sqrt{23}}\right)$$

Q No. 24 $I = \sqrt{a^2 - x^2}dx$

Put $x = asin\theta \Rightarrow dx = acos\theta d\theta$

$$I = \int \sqrt{a^2 - a^2sin^2\theta}\, acos\theta\, d\theta$$

$$I = \int a\sqrt{1 - sin^2\theta}\, acos\theta\, d\theta$$

$$I = a^2\int cos^2\theta d\theta$$

$$I = \frac{a^2}{2}\int (1 + cos2\theta)d\theta \qquad as\ \ cos^2\theta = \frac{1 + cos2\theta}{2}$$

$$I = \frac{a^2}{2}(\theta + \frac{sin2\theta}{2})$$

$$I = \frac{a^2}{2}(\theta + \frac{2sin\theta cos\theta}{2})$$

$$I = \frac{a^2}{2}(\theta + sin\theta\sqrt{1 - sin^2\theta})$$

Substitution returned:

$$I = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right)$$

$$I = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{\frac{a^2 - x^2}{a^2}}\right)$$

$$I = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}.\frac{x}{a}.\frac{\sqrt{a^2 - x^2}}{a}$$

$$I = \frac{x}{2}.\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

Q No. 25 $I = \int (2x + 3)\sqrt{2x + 1}\, dx$

$$I = \int (2x + 1 + 2)\sqrt{2x + 1}dx$$

$$I = \int (2x + 1)\sqrt{2x + 1}dx + 2\int \sqrt{2x + 1}\, dx$$

$$I = \int (2x + 1)^{1+\frac{1}{2}} dx + 2\int (2x + 1)^{\frac{1}{2}} dx$$

$$I = \frac{1}{2}\int (2x + 1)^{\frac{3}{2}}.2dx + \int (2x + 1)^{\frac{1}{2}} 2.dx$$

$$I = \frac{1}{2}.\frac{(2x + 1)^{\frac{3}{2}+1}}{\frac{3}{2} + 1} + \frac{(2x + 1)^{\frac{1}{2}+1}}{\frac{1}{2} + 1}$$

Q No. 26 $I = \int (1 + x^2)^{-\frac{3}{2}} dx$

$Put\ \ x = tan\theta \ \ \Rightarrow \ \ dx = sec^2\theta d\theta$

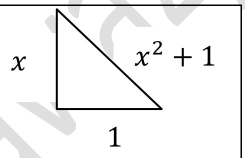
$$I = \int (1 + tan^2\theta)^{-\frac{3}{2}}.sec^2\theta\, d\theta$$

$$I = \int (sec^2\theta)^{-\frac{3}{2}}.sec^2\theta\, d\theta$$

$$I = \int \frac{sec^2\theta}{sec^3\theta}d\theta \ \ = \int cos\theta d\theta \ \ = sin\theta$$

In right triangle :

$tan\theta = \frac{x}{1}$



By Pythagorean's theorem we can find the Hyp. So

$sin\theta = \frac{x}{x^2+1}$

Hence $I = \frac{x}{x^2+1}$

Q No. 27 $I = \int \frac{x^2}{\sqrt{x^2+1}}dx$

$$I = \int \frac{x^2 + 1 - 1}{\sqrt{x^2 + 1}}dx \ \ = \int \left(\frac{x^2 + 1}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x^2 + 1}}\right)dx$$

$$I = \int \sqrt{x^2 + 1}dx - \int \frac{dx}{\sqrt{x^2 + 1}}$$

$$I = \left[\frac{x}{2}\sqrt{x^2 + 1} + \frac{1}{2}\sinh^{-1}x\right] - \sinh^{-1}x$$

$$I = \frac{x}{2}\sqrt{x^2 + 1} - \frac{1}{2}\sinh^{-1}x$$

Q No. 28 $I = \int (2x + 4)\sqrt{2x^2 + 3x + 1}dx$

$$I = \frac{1}{2}\int (2x^2 + 3x + 1)^{\frac{1}{2}}.(4x + 8)dx$$

$$I = \frac{1}{2}\int (2x^2 + 3x + 1)^{\frac{1}{2}}.(4x + 3 + 5)dx$$

$$I = \frac{1}{2}\int (2x^2 + 3x + 1)^{\frac{1}{2}}.(4x + 3)dx + \frac{1}{2}\int (2x^2 + 3x + 1)^{\frac{1}{2}}.5dx$$

$$I = \frac{1}{2}\frac{(2x^2 + 3x + 1)^{\frac{1}{2}+1}}{\frac{1}{2} + 1} + \frac{5}{2}.\sqrt{2}\int \left(x^2 + \frac{3}{2}x + \frac{1}{2}\right)^{\frac{1}{2}} dx$$

$$I = \frac{1}{3}(2x^2 + 3x + 1)^{\frac{3}{2}} + \frac{5}{\sqrt{2}}\int \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}\, dx$$

Completing square of

$$x^2+\frac{3}{2}x+\frac{1}{2}$$

$$=(x)^2+2\left(\frac{3}{4}\right)(x)+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+\frac{1}{2}$$

$$=\left(x+\frac{3}{4}\right)^2-\frac{9}{16}+\frac{1}{2}$$

$$=\left(x+\frac{3}{4}\right)^2-\frac{9}{16}+\frac{8}{16}$$

$$=\left(x+\frac{3}{4}\right)^2-\frac{1}{16}$$

$$=\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2$$

Hence,

$$I=\frac{1}{3}(2x^2+3x+1)^{\frac{3}{2}}$$

$$+\frac{5}{\sqrt{2}}\left[\frac{x+\frac{3}{4}}{2}\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}-\frac{(1/4)^2}{2}\cosh^{-1}\frac{x+\frac{3}{4}}{\frac{1}{4}}\right]$$

$$I=\frac{1}{3}(2x^2+3x+1)^{\frac{3}{2}}$$

$$+\frac{5}{\sqrt{2}}\left[\frac{4x+3}{8}.\frac{\sqrt{(4x+3)^2-(1)^2}}{4}-\frac{1}{32}\cosh^{-1}(4x+3)\right]$$

$$I=\frac{1}{3}(2x^2+3x+1)^{\frac{3}{2}}$$

$$+\frac{5}{32\sqrt{2}}\left[4x+\sqrt{(4x+3)^2-1}-\cosh^{-1}(4x+3)\right]$$

Q No. 29 $\quad I=\int\frac{dx}{3sinx+4cosx}$

Let $3=rsint$ and $4=rcost$

Squaring and adding, we get,	Dividing , we get
$3^2+4^2=r^2sin^2t+r^2cos^2t$ $25=r^2$ $r=5$	$\frac{rsint}{rcost}=\frac{3}{4}$ $t=\tan^{-1}(\frac{3}{4})$

$$I=\int\frac{dx}{rsintsinx+rcostcosx}$$

$$I=\frac{1}{r}\int\frac{dx}{\cos(x-t)}$$

$$I=\frac{1}{r}\int\sec(x-t)\,dx$$

$$I=\frac{1}{r}ln|\sec(x-t)+\tan(x-t)|$$

$$I=\frac{1}{5}ln\left|\sec\left(x-\tan^{-1}\frac{3}{4}\right)+tan\left(x-\tan^{-1}\frac{3}{4}\right)\right|$$

Q No. 30 $\quad I=\int\frac{tanxdx}{cosx+secx}$

$$I=\int\frac{\frac{sinx}{cosx}}{cosx+\frac{1}{cosx}}dx$$

$$I=\int\frac{sinx}{\cos^2x+1}dx$$

$$I=-\int\frac{-sinx}{\cos^2x+1}dx$$

$$I=\tan^{-1}(cosx)$$

Q No. 31 $\quad I=\int\frac{dx}{sin(x-a)sin(x-b)}$

$$1=\frac{\sin(a-b)}{\sin(a-b)}=\frac{\sin(a-b+x-x)}{\sin(a-b)}$$

$$=\frac{\sin(x-b-x+a)}{\sin(a-b)}$$

$$=\frac{\sin[(x-b)-(x-a)]}{\sin(a-b)}$$

$$=\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(a-b)}$$

So,

$$I=\frac{1}{\sin(a-b)}$$

$$\int\left(\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right)$$

$$I=\frac{1}{\sin(a-b)}\int\left[\frac{\cos(x-a)}{\sin(x-a)}-\frac{\cos(x-b)}{\sin(x-b)}\right]dx$$

$$I=\frac{1}{\sin(a-b)}[ln\sin(x-a)-ln\sin(x-b)]$$

$$I=\frac{1}{\sin(a-b)}ln\frac{\sin(x-a)}{\sin(x-b)}$$

Q No. 32 $I = \int tanx ln(secx)\, dx$

Put $lnsecx = z \quad \Rightarrow \quad dz = \frac{1}{secx}.secx.tanx.dx$

$\Rightarrow \quad dz = \text{tanxdx}$

$I = zdz = \frac{z^2}{2} = \frac{(lnsecx)^2}{2}$

Q No. 33 $I = \int \frac{dx}{(3tanx+1)cos^2x}$

$I = \int \frac{\sec^2 xdx}{3tanx+1} \quad = \frac{1}{3}\int \frac{3\sec^2 xdx}{3tanx+1} \quad = \frac{1}{3}\ln(3tanx+1)$

Q No. 34 $I = \int e^{sinx} cosx\, dx$

Put $sinx = z \quad \Rightarrow \quad cosxdx = dz$

$I = \int e^z dz = e^z = e^{sinx}$

Q No. 35 $I = \int \sqrt{1+3cos^2x}\, sin2xdx$

Put $\cos^2 x = z \quad \Rightarrow \quad 2cosx(-sinx)dx = dz$

$\Rightarrow \quad -2sinx.cosxdx = dz \quad or \quad sin2xdx = -dz$

$I = -\int (1+3z)^{\frac{1}{2}} dz$

$I = -\frac{1}{3}\int (1+3z)^{\frac{1}{2}}.3dz$

$I = -\frac{1}{3}.\frac{(1+3z)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$I = -\frac{2}{9}(1+3z)^{\frac{3}{2}}$

$I = -\frac{2}{9}(1+3\cos^2 x)^{\frac{3}{2}}$

Q No. 36 $I = \int \frac{sin2xdx}{\sqrt{1+cos^2x}}$

Put $\cos^2 x = z \quad \Rightarrow \quad 2cosx(-sinx)dx = dz$

$\Rightarrow \quad -2sinx.cosxdx = dz \quad or \quad sin2xdx = -dz$

$I = -\int (1+z)^{\frac{-1}{2}} dz$

$I = -\frac{(1+z)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}$

$I = -2\sqrt{1+\cos^2 x}$

Q No. 37 $I = \int \frac{dx}{2sin^2x+3cos^2x}$

$Divide\ N^r and\ D^r by \cos^2 x$

$I = \int \frac{\sec^2 xdx}{2\tan^2 x + 3}$

Put $\tan x = z \qquad \sec^2 xdx = dz$

$I = \int \frac{dz}{2z^2+3} \quad = \frac{1}{2}\int \frac{dz}{z^2+\frac{3}{2}}$

$I = \frac{1}{2}.\sqrt{\frac{2}{3}}\tan^{-1}\frac{\sqrt{2}z}{\sqrt{3}} \qquad as\ (\frac{1}{a}\tan^{-1}\frac{x}{a})$

Q No. 38 $I = \int \frac{1}{\sqrt{x}} sec\sqrt{x} tan\sqrt{x}\, dx$

Put $\sqrt{x} = z \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dz \quad \Rightarrow \quad \frac{dx}{\sqrt{x}} = 2dz$

$I = \int secztanz.2dz = 2secz = 2sec\sqrt{x}$

Q No. 39 $I = \int [\pi^{sinx} + (sinx)^{\pi}]cosxdx$

Put $sinx = z \quad \Rightarrow cosxdx = dz$

$I = \int \pi^z dz + \int z^{\pi} dz$

$I = \frac{\pi^z}{ln\pi} + \frac{z^{\pi+1}}{\pi+1}$

$I = \frac{\pi^{sinx}}{ln\pi} + \frac{(sinx)^{\pi+1}}{\pi+1}$

Q No. 40 $I = \int \frac{cosxdx}{3sinx+4\sqrt{sinx}}$

Put $\sqrt{sinx} = z \quad \Rightarrow \quad sinx = z^2 \quad \Rightarrow \quad cosxdx = 2zdz$

$I = \int \frac{2zdz}{3z^2+4z} = 2\int \frac{dz}{3z+4} \quad = \frac{2}{3}\int \frac{3dz}{3z+4}$

$I = \frac{2}{3}\ln(3z+4)$

$I = \frac{2}{3}\ln(3\sqrt{sinx}+4)$

Techniques of Integration

EXERCISE 4.3

Integration By Parts means 4 brackets in such a way that:

$$(\ )\ (\ )\ -\ \int (\ )(\ )dx$$

$(F1)\ (integral\ of\ F2)$

$-\ \int (derivative\ of\ F1)(integral\ of\ F2)dx$

Or

$$(F1)\left(\int F2.dx\right)$$

$$-\int (F1')\left(\int F2.dx\right)dx$$

Q No. 1 $I = \int xsec^2xdx$

Applying By Parts

$$I = (x)(tanx) - \int (1)(tanx)dx$$

$$I = xtanx - \int tanxdx$$

$$I = xtanx - \ln(secx) + c$$

Q No. 2 $I = \int xcsc^2xdx$

$$I = (x)(-cotx) - \int (1)(-cotx)dx$$

$$I = -xcotx + \int cotxdx$$

$$I = -xcotx + \ln(sinx) + c$$

Q No. 3 $I = \int x^n lnxdx$

$$I = (lnx)\left(\frac{x^{n+1}}{n+1}\right) - \int \left(\frac{1}{x}\right)\left(\frac{x^{n+1}}{n+1}\right)dx$$

$$I = (lnx)\left(\frac{x^{n+1}}{n+1}\right) - \frac{1}{n+1}\int x^n dx$$

$$I = \frac{x^{n+1}(lnx)}{n+1} - \frac{1}{n+1}.\frac{x^{n+1}}{n+1}$$

$$I = \frac{x^{n+1}(lnx)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$$

Q No. 4 $I = \int x^2 \tan^{-1} x\,dx$

$$I = (\tan^{-1} x)\left(\frac{x^{2+1}}{2+1}\right) - \int \left(\frac{1}{x^2+1}\right)\left(\frac{x^{2+1}}{2+1}\right)dx$$

$$I = (\tan^{-1} x)\left(\frac{x^3}{3}\right) - \frac{1}{3}\int \left(\frac{x^3}{x^2+1}\right)dx$$

In $\frac{x^3}{x^2+1}$ we use long division and get $x - \frac{x}{x^2+1}$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int \left(x - \frac{x}{x^2+1}\right)dx$$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int xdx + \frac{1}{3}\int \left(\frac{x}{x^2+1}\right)dx$$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}.\frac{x^2}{2} + \frac{1}{3}.\frac{1}{2}\int \left(\frac{2x}{x^2+1}\right)dx$$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{x^2}{6} + \frac{1}{6}\ln(x^2+1) + c$$

Q No.5 $I = \int sec^3xdx$

$$I = \int secx.sec^2xdx$$

$$I = (secx)(tanx) - \int (secx.tanx)(tanx)dx$$

$$I = secxtanx - \int secx.(tan^2x)dx$$

$$I = secxtanx - \int secx.(sec^2x - 1)dx$$

$$I = secxtanx - \int sec^3xdx + \int secxdx$$

Here we have,

$$I = secxtanx - I + \ln(secx + tanx)$$

Prepared by: **Aqeel Nawaz** aqeel_755@yahoo.com 03349587500

$$I+I=secxtanx+\ln(secx+tanx)$$

$$2I=secxtanx+\ln(secx+tanx)$$

$$I=\frac{1}{2}secxtanx+\frac{1}{2}\ln(secx+tanx)+c$$

Q No. 6 $I=\int csc^3xdx$

$$I=\int cscx.csc^2xdx$$

$$I=(cscx)(-cotx)-\int(-cscxcotx)(-cotx)dx$$

$$I=-cscxcotx-\int cscx(cot^2x)dx$$

$$I=-cscxcotx-\int cscx(csc^2x-1)dx$$

$$I=-cscxcotx-\int csc^3xdx+\int cscxdx$$

$$I=cscxcotx-I+\ln(cscx-cotx)$$

$$I+I=-cscxcotx+\ln(cscx-cotx)$$

$$2I=-cscxcotx+\ln(cscx-cotx)$$

$$I=\frac{-1}{2}cscxcotx+\frac{1}{2}\ln(cscx-cotx)+c$$

Q No. 7 $I=\int\frac{x-sinx}{1-cosx}dx$

In trigonometry we write:

$\sin^2\left(\frac{x}{2}\right)=\frac{1-cosx}{2}$ and $sinx=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

So,

$$I=\int\frac{x-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}dx$$

$$I=\int\frac{x}{2\sin^2\left(\frac{x}{2}\right)}dx-\int\frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}dx$$

$$I=\frac{1}{2}\int xcsc^2(\frac{x}{2})dx-\int\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}dx$$

$$I=\frac{1}{2}\int xcsc^2(\frac{x}{2})dx-\int\cot(\frac{x}{2})dx$$

$$I=\frac{1}{2}\int xcsc^2(\frac{x}{2})dx-2\ln(sin\frac{x}{2})\qquad \text{------(1)}$$

$$I_1=\int xcsc^2\left(\frac{x}{2}\right)dx$$

$$I_1=(x)\left(-2\cot\frac{x}{2}\right)-\int(1)\left(-2\cot\frac{x}{2}\right)dx$$

$$I_1=-2x\cot\frac{x}{2}+2\int\cot\frac{x}{2}dx$$

$$I_1=-2x\cot\frac{x}{2}+2.2\ln(sinx)$$

$$I_1=-2x\cot\frac{x}{2}+4\ln(sinx)$$

Hence,

$$I=\frac{1}{2}\left[-2x\cot\frac{x}{2}+4\ln\left(sin\frac{x}{2}\right)\right]-2\ln(sin\frac{x}{2})$$

$$I=-x\cot\frac{x}{2}+2\ln\left(sin\frac{x}{2}\right)-2\ln(sin\frac{x}{2})+c$$

$$I=-x\cot\frac{x}{2}+c$$

Alternative method (after step 4):

$$I=\frac{1}{2}\int xcsc^2(\frac{x}{2})dx-\int\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}dx$$

$$I=\frac{1}{2}\int xcsc^2(\frac{x}{2})dx-\int\cot(\frac{x}{2})dx$$

Applying by parts on first

$$I=\frac{1}{2}\left[(x)\left(\frac{-\cot\frac{x}{2}}{\frac{1}{2}}\right)-\int(1)\left(\frac{-\cot\frac{x}{2}}{\frac{1}{2}}\right)dx\right]-\int\cot(\frac{x}{2})dx$$

$$I=-x\cot\frac{x}{2}+\int\cot(\frac{x}{2})dx-\int\cot(\frac{x}{2})dx$$

$$I=-xcot\frac{x}{2}+c$$

Q No. 8 $I=\int x\sin^{-1}x\,dx$

$$I=(\sin^{-1}x)\left(\frac{x^2}{2}\right)-\int\left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{x^2}{2}\right)dx$$

$$I=\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left[\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right]dx$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left[\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right]dx$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]-\frac{1}{2}\sin^{-1}x$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x-\frac{1}{2}\sin^{-1}x$$

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}-\frac{1}{4}\sin^{-1}x+c$$

Q No. 9 $I=\int x^3\sqrt{x^2+1}\,dx$

$$I=\int x^2.x\sqrt{x^2+1}\,dx$$

$$I=\int(x^2+1-1).x\sqrt{x^2+1}\,dx$$

$I=\int(x^2+1).x\sqrt{x^2+1}\,dx-\int x\sqrt{x^2+1}\,dx$

$I=\int(x^2+1)^{1+\frac{1}{2}}.x\,dx-\int(x^2+1)^{\frac{1}{2}}x\,dx$

$I=\frac{1}{2}\int(x^2+1)^{\frac{3}{2}}.2x\,dx-\frac{1}{2}\int(x^2+1)^{\frac{1}{2}}.2x\,dx$

$I=\frac{1}{2}.\frac{(x^2+1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}-\frac{1}{2}.\frac{(x^2+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$

$I=\frac{1}{2}.\frac{(x^2+1)^{\frac{5}{2}}}{\frac{5}{2}}-\frac{1}{2}.\frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}}+c$

$I=\frac{(x^2+1)^{\frac{5}{2}}}{5}-\frac{(x^2+1)^{\frac{3}{2}}}{3}+c$

Alternative method of QNo. 9 is at the end of the exercise (page9)

Q No. 10 $I=\int e^x\frac{1+xlnx}{x}dx$

$I=\int e^x\frac{1}{x}dx+\int e^x\,lnxdx$

Applying by parts on first angle,

$I=(e^x)(lnx)-\int(e^x)(lnx)dx+\int e^x\,lnxdx$

$I=e^xlnx+c$

Q No. 11 $I=\int e^x\frac{1-sinx}{1-cosx}dx$

$I=\int e^x.\left(\frac{1-sinx}{1-cosx}\times\frac{1+cosx}{1+cosx}\right)dx$

$I=\int e^x.\left(\frac{1-sinx+cosx-sinxcosx}{1-cos^2x}\right)dx$

$I=\int e^x.\left(\frac{1-sinx+cosx-sinxcosx}{sin^2x}\right)dx$

$I=\int e^x\left(\frac{1}{sin^2x}-\frac{sinx}{sin^2x}+\frac{cosx}{sin^2x}-\frac{sinxcosx}{sin^2x}\right)dx$

$I=\int e^x\,(csc^2x-cscx+cscxcotx-cotx)dx$

$I=\int e^xcsc^2x\,dx-\int e^x\,cscx\,dx+\int e^xcscxcotx\,dx-\int e^x\,cotx\,dx$

Hence our integral becomes as follows,

We will apply By Parts technique upon 1st and 3rd integral:

$I_1=\int e^xcsc^2x\,dx$

$=(e^x)(-cotx)-\int(e^x)\,(-cotx)\,dx$

$=-e^xcotx+\int e^xcotxdx$ ------- (1)

And

$I_3=\int e^xcscxcotx\,dx$

$=(e^x)(-cscx)-\int(e^x)\,(-cscx)\,dx$

$=-e^xcscx+\int e^xcscxdx$ ------- (2)

Putting values in I we get:

$$I=-e^xcotx-e^xcscx+c$$

Q No. 12 $I=\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

$Put\ \tan^{-1}\sqrt{\frac{1-x}{1+x}}=z---\rightarrow(1)$

$\sqrt{\frac{1-x}{1+x}}=tanz$

$\frac{1-x}{1+x}=tan^2z$

$(1-x)=tan^2z(1+x)$

$1-x=tan^2z+xtan^2z$

$1-tan^2z=x+xtan^2z$

$1-tan^2z=x(1+tan^2z)$

Multiply D^r and N^r by cos^2z

$\frac{1-tan^2z}{1+tan^2z}=x\quad\Rightarrow\frac{cos^2z-sin^2z}{cos^2z+sin^2z}=x$

$\Rightarrow\frac{cos2z}{1}=x\ \Rightarrow\ cos2z=x\quad---\rightarrow(2)$

Diff. w.r.t x

$-2sin2zdz=dx$

$$I = \int z.(-2sin2z)dz$$

$$I = -2\int zsin2z\,dz$$

Applying integration by parts

$$I = -2(z)\left(-\frac{cos2z}{2}\right) + 2\int (1)\left(-\frac{cos2z}{2}\right)dz$$

$$I = zcos2z - \int cos2zdz$$

$$I = zcos2z - \frac{sin2z}{2} + c$$

Re-back substitution, using eqs. (1) & (2)

$$I = \tan^{-1}\sqrt{\frac{1-x}{1+x}}.x - \frac{\sqrt{1-x^2}}{2} + c$$

Q No. 13 $\quad I = \int \sin^{-1}\sqrt{\frac{x}{x+a}}dx$

$$Put\ \sin^{-1}\sqrt{\frac{x}{x+a}} = z - - - - - \to (1)$$

$$\sqrt{\frac{x}{x+a}} = sinz$$

$$\frac{x}{x+a} = sin^2z$$

$$x = sin^2z(x+a)$$

$$x = xsin^2z + asin^2z$$

$$x - xsin^2z = asin^2z$$

$$x(1 - sin^2z) = asin^2z$$

$$xcos^2z = asin^2z$$

$$x = atan^2z - - - \to (2)$$

$$dx = 2atanz\,sec^2z\,dz$$

Hence our integral become,

$$I = \int z\,(2atanz\,sec^2z)\,dz$$

$$I = 2a\int ztanz\,sec^2z\,dz$$

$$I = 2a(z)\left(\frac{tan^2z}{2}\right) - 2a\int (1)\left(\frac{tan^2z}{2}\right)dz$$

$$I = aztan^2z - a\int tan^2z\,dz$$

$$I = aztan^2z - a\int (sec^2z - 1)\,dz$$

$$I = aztan^2z - a\int sec^2zdz + a\int dz$$

$$I = aztan^2z - atanz + az + c$$

Re-back substitution, using eqs. (1) and (2)

$$I = a\frac{x}{a}\sin^{-1}\sqrt{\frac{x}{x+a}} - a\sqrt{\frac{x}{a}} + a\sin^{-1}\sqrt{\frac{x}{x+a}} + c$$

$$I = a\sin^{-1}\sqrt{\frac{x}{x+a}} - \sqrt{ax} + a\sin^{-1}\sqrt{\frac{x}{x+a}} + c$$

$$I = 2a\sin^{-1}\sqrt{\frac{x}{x+a}} - \sqrt{ax} + c$$

Q No. 14 $\quad I = \int e^{ax}\sin(bx+c)dx$

Using by parts formula,

$$I = (e^{ax})\left(\frac{-\cos(bx+c)}{b}\right) - \int (ae^{ax})\left(\frac{-\cos(bx+c)}{b}\right)dx$$

$$I = \frac{-e^{ax}\cos(bx+c)}{b} + \frac{a}{b}\int e^{ax}\cos(bx+c)\,dx$$

Using by parts formula,

$$I = -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b}(e^{ax})\left(\frac{\sin(bx+c)}{b}\right) - \frac{a}{b}\int (ae^{ax})\left(\frac{\sin(bx+c)}{b}\right)dx$$

$$I = -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}\int e^{ax}\sin(bx+c)\,dx$$

$$I = -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}I$$

$$I + \frac{a^2}{b^2}I = -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$\left(1+\frac{a^2}{b^2}\right)I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$\left(\frac{a^2+b^2}{b^2}\right)I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$I=-\frac{b}{a^2+b^2}e^{ax}\cos(bx+c)+\frac{1}{a^2+b^2}e^{ax}\sin(bx+c)$$

Q No. 15 $\boldsymbol{I=\int ln\left(x+\sqrt{1+x^2}\right)dx}$

$$I=\int 1.ln(x+\sqrt{1+x^2})dx$$

$$I=\left(ln(x+\sqrt{1+x^2})\right)(x)-\int\left(\frac{1+\frac{2x}{2\sqrt{1+x^2}}}{(x+\sqrt{1+x^2})}\right)(x)dx$$

$$I=\left(ln(x+\sqrt{1+x^2})\right)(x)-\int\left(\frac{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{(x+\sqrt{1+x^2})}\right)(x)dx$$

$$I=\left(ln(x+\sqrt{1+x^2})\right)(x)-\int\left(\frac{x}{\sqrt{1+x^2}}\right)dx$$

$$I=xln(x+\sqrt{1+x^2})-\int(1+x^2)^{-\frac{1}{2}}xdx$$

$$I=xln(x+\sqrt{1+x^2})-\frac{1}{2}\int(1+x^2)^{-\frac{1}{2}}.2xdx$$

$$I=xln\left(x+\sqrt{1+x^2}\right)-\frac{1}{2}.\frac{(1+x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$I=xln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+c$$

Q No. 16 $\boldsymbol{I=\int\frac{x^2+1}{(x+1)^2}e^x\,dx}$

$$I=\int\frac{x^2+1}{(x+1)^2}e^x\,dx$$

$$I=\int\frac{x^2+1+2x-2x}{(x+1)^2}e^x\,dx$$

$$I=\int\frac{(x+1)^2-2x}{(x+1)^2}e^x\,dx$$

$$I=\int e^xdx-\int\frac{2xe^x}{(x+1)^2}dx$$

$$I=e^x-\int\frac{2xe^x}{x^2+2x+1}dx$$

$$I=e^x-\int\frac{(2x+2-2)e^x}{(x+1)^2}dx$$

$$I=e^x-\int\frac{(2x+2)e^x}{(x+1)^2}dx+2\int\frac{e^x}{(x+1)^2}dx$$

$$I=e^x-2\int\frac{(x+1)e^x}{(x+1)^2}dx+2\int\frac{e^x}{(x+1)^2}dx$$

$$I=e^x-2\int\frac{e^x}{x+1}dx+2\int\frac{e^x}{(x+1)^2}dx$$

Integrating first integral by parts,

$$I=e^x-2\left(\frac{1}{x+1}\right)(e^x)+2\int\frac{(-1)}{(x+1)^2}.(e^x)dx+2\int\frac{e^x}{(x+1)^2}dx$$

$$I=e^x-\frac{2e^x}{x+1}-2\int\frac{e^x}{(x+1)^2}dx+2\int\frac{e^x}{(x+1)^2}dx$$

$$I=e^x-\frac{2e^x}{x+1}+c$$

Q No. 17 $\boldsymbol{I=\int cos(lnx)\,dx}$

$$I=(\cos(lnx))(x)-\int\frac{-\sin(lnx)}{x}(x)dx$$

$$I = x.\cos(lnx) + \int \sin(lnx)\,dx$$

Integrating again by parts,

$$I = x.\cos(lnx) + (sinlnx)(x) - \int \cos\frac{(lnx)}{x}(x)\,dx$$

$$I = x[\cos(lnx) + \sin(lnx)] - \int \cos(lnx)\,dx$$

$$I = x[\cos(lnx) + \sin(lnx)] - I$$

$$2I = x[\cos(lnx) + \sin(lnx)]$$

$$I = \frac{x}{2}[\cos(lnx) + \sin(lnx)]$$

Q No. 18 $\boldsymbol{I = \int \sqrt{x}\, e^{-\sqrt{x}} dx}$

$$Put\ \sqrt{x} = z \ \Rightarrow x = z^2 \ \Rightarrow \ dx = 2zdz$$

$$I = \int ze^{-z}.2zdz \ = 2\int z^2e^{-z}dz$$

Integrating by parts

$$I = 2(z^2)(-e^{-z}) - 2\int 2z.(-e^{-z})dz$$

$$I = -2z^2e^{-z} + 4\int ze^{-z}dz$$

Integrating by parts again

$$I = -2z^2e^{-z} + 4(z)(-e^{-z}) - 4\int (1)(-e^{-z})dz$$

$$I = -2z^2e^{-z} - 4ze^{-z} + 4\int e^{-z}dz$$

$$I = -2z^2e^{-z} - 4ze^{-z} - 4e^{-z} + c$$

Hence

$$I = -2xe^{-\sqrt{x}} - 4\sqrt{x}e^{-\sqrt{x}} - 4e^{-\sqrt{x}} + c$$

Q No. 19 $\boldsymbol{I = \int x^3e^{2x}\,dx}$

$$I = \int x^3e^{2x}dx$$

Integrating by parts,

$$I = (x^3)\left(\frac{e^{2x}}{2}\right) - \int (3x^2)\left(\frac{e^{2x}}{2}\right)dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3}{2}\int x^2e^{2x}dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3}{2}(x^2)\left(\frac{e^{2x}}{2}\right) + \frac{3}{2}\int 2x.\frac{e^{2x}}{2}dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\int xe^{2x}dx$$

$$= \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3}{2}(x)\left(\frac{e^{2x}}{2}\right) - \frac{3}{2}\int (1)\left(\frac{e^{2x}}{2}\right)dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3x}{4}e^{2x} - \frac{3}{4}\int e^{2x}dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3x}{4}e^{2x} - \frac{3}{4}.\frac{e^{2x}}{2} + c$$

Q No. 20 $\boldsymbol{I = \int x^5e^{x^3}dx}$

$$I = \int x^3\,x^2e^{x^3}dx$$

$$Put\ x^3 = z \qquad \Rightarrow \ 3x^2dx = dz \qquad \Rightarrow x^2dx = \frac{dz}{3}$$

Hence our integral becomes:

$$I = \int ze^z\frac{dz}{3} \quad = \frac{1}{3}\int ze^z\,dz$$

Apllying by parts,

$$I = \frac{1}{3}(z)(e^z) - \frac{1}{3}\int (1)(e^z)dx$$

$$I = \frac{ze^z}{3} - \frac{1}{3}\int e^z\,dz$$

$$I = \frac{ze^z}{3} - \frac{e^z}{3} + c$$

$$I = \frac{x^3e^{x^3}}{3} - \frac{e^{x^3}}{3} + c$$

Q No. 21 Show that

$$\int x^n tan^{-1}x\,dx = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}dx$$

Hence evaluate, $\int x^3 \tan^{-1} x\,dx$

Let,

$$I = \int x^n tan^{-1}x\,dx$$

Integrating by parts,

$$I = (tan^{-1}x)\left(\frac{x^{n+1}}{n+1}\right) - \int \left(\frac{1}{1+x^2}\right)\left(\frac{x^{n+1}}{n+1}\right)dx$$

$$I = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}dx$$

As required.

Now, put n=3

$$I = \frac{x^{3+1}}{3+1}\tan^{-1}x - \frac{1}{3+1}\int \frac{x^{3+1}}{1+x^2}dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{1+x^2}dx$$

By long division we get,

$$\frac{x^4}{1+x^2} = x^2 - 1 + \frac{1}{1+x^2}$$

So I becomes,

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int\left(x^2 - 1 + \frac{1}{1+x^2}\right)dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int x^2\,dx + \frac{1}{4}\int dx - \frac{1}{4}\int \frac{dx}{1+x^2}$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}.\frac{x^3}{3} + \frac{x}{4} - \frac{1}{4}\tan^{-1}x + c$$

Q No. 22 Find a reduction formula for $\int x^n e^{ax}dx$ and apply it to evaluate $\int x^3 e^{ax}dx$.

$$I = \int x^n e^{ax}dx$$

Applying by parts formula,

$$I = (x^n)\left(\frac{e^{ax}}{a}\right) - \int (nx^{n-1})\left(\frac{e^{ax}}{a}\right)dx$$

$$I = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

Which is the required reduction formula.

Now, put $n = 3$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^{3-1}e^{ax}\,dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\,dx$$

Again put $n - 2$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[\frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x^{2-1}e^{ax}\,dx\right]$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}\int x\,e^{ax}\,dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}(x)\left(\frac{e^{ax}}{a}\right) - \frac{6}{a^2}\int (1)\left(\frac{e^{ax}}{a}\right)dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6xe^{ax}}{a^3} - \frac{6}{a^3}\int e^{ax}dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6xe^{ax}}{a^3} - \frac{6}{a^3}.\frac{e^{ax}}{a} + c$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6xe^{ax}}{a^3} - \frac{6e^{ax}}{a^3} + c$$

Q No. 23 Find a reduction formula for $\int \boldsymbol{sin^n x dx}$ and $\int \boldsymbol{cos^n x dx}$ where n is a positive integer.

$$I = \int sin^n x dx$$

We separate a single powerof sinx. As follows:

$$I = \int sin^{n-1}x\ sinxdx$$

Applying by parts formula

$$I = (sin^{n-1}x)(-cosx) - \int (n-1)(sin^{n-2}x.cosx)(-cosx)dx$$

$$I = -cosx.sin^{n-1}x + (n-1)\int sin^{n-2}x.cos^2x\ dx$$

$$I = -cosx.sin^{n-1}x + (n-1)\int sin^{n-2}x.(1-sin^2x)\ dx$$

$$= -cosx.sin^{n-1}x + (n-1)\int sin^{n-2}xdx - (n-1)\int sin^n x\ dx$$

$$I = -cosxsin^{n-1}x + (n-1)\int sin^{n-2}xdx - (n-1)I$$

$$I + (n-1)I = -cosxsin^{n-1}x + (n-1)\int sin^{n-2}xdx$$

$$(1+n-1)I = -cosxsin^{n-1}x + (n-1)\int sin^{n-2}xdx$$

$$nI = -cosxsin^{n-1}x + (n-1)\int sin^{n-2}xdx$$

$$I = \frac{-cosxsin^{n-1}x}{n} + \frac{(n-1)}{n}\int sin^{n-2}xdx$$

And,

$$I = \int cos^n xdx$$

We separate a single powerof sinx. As follows:

$$I = \int cos^{n-1}x\ cosxdx$$

Applying by parts formula

$$I = (cos^{n-1}x)(sinx) - \int (n-1)(cos^{n-2}x)(-sinx)(sinx)dx$$

$$I = sinx.cos^{n-1}x + (n-1)\int cos^{n-2}x.sin^2x\ dx$$

$$I = sinx.cos^{n-1}x + (n-1)\int cos^{n-2}x.(1-cos^2x)\ dx$$

$$= sinx.cos^{n-1}x + (n-1)\int cos^{n-2}xdx - (n-1)\int cos^n x\ dx$$

$$I = sinx.cos^{n-1}x + (n-1)\int cos^{n-2}xdx - (n-1)I$$

$$I + (n-1)I = sinx.cos^{n-1}x + (n-1)\int cos^{n-2}xdx$$

$$(1+n-1)I = sinx.cos^{n-1}x + (n-1)\int cos^{n-2}xdx$$

$$nI = sinx.cos^{n-1}x + (n-1)\int cos^{n-2}xdx$$

$$I = \frac{sinx.cos^{n-1}x}{n} + \left(\frac{n-1}{n}\right)\int cos^{n-2}xdx$$

Q No. 24 Find a reduction formula for $\int \boldsymbol{x^n sinax\ dx}$, $\boldsymbol{where\ n > 1}$ is an integer. Hence evaluate $\int \boldsymbol{x^4 sinax\ dx}$.

$$I = \int x^n sinax$$

Integrating by parts,

$$I = (x^n)\left(\frac{-cosax}{a}\right) - \int (nx^{n-1})\left(\frac{-cosax}{a}\right)dx$$

$$I = \frac{-x^n cosax}{a} + \frac{n}{a}\int x^{n-1}cosaxdx$$

Again by parts,

$$I = \frac{-x^n cosax}{a} +$$

$$\frac{n}{a}(x^{n-1})\left(\frac{sinax}{a}\right) - \frac{n}{a}\int (n-1)x^{n-2}\left(\frac{sinax}{a}\right)dx$$

$$I=\frac{-x^n cosax}{a}+\frac{nx^{n-1}sinax}{a^2}-\frac{n(n-1)}{a^2}\int x^{n-2}sinaxdx$$

Which is the required reduction formula,

Put n=4 & a=4

$$I=\frac{-x^4cos4x}{4}+\frac{4x^3sin4x}{16}-\frac{4.3}{16}\int x^2sin4xdx$$

Now put n=2 and a=4,

$$I=\frac{-x^4cos4x}{4}+\frac{4x^3sin4x}{16}-\frac{4.3}{16}$$

$$\left[\frac{-x^2cos4x}{2}+\frac{2xcos4x}{4}-\frac{2}{4}\int sin4xdx\right]$$

$$I=\frac{-x^4cos4x}{4}+\frac{4x^3sin4x}{16}$$
$$-\frac{3}{4}\left[\frac{-x^2cos4x}{2}+\frac{2xcos4x}{4}-\frac{2}{4}\frac{cos4x}{4}\right]$$

$$I=\frac{-x^4cos4x}{4}+\frac{4x^3sin4x}{16}+\frac{3x^2cos4x}{8}+\frac{3xcos4x}{8}$$
$$-\frac{3cos4x}{32}+c$$

Q No. 25 Find a reduction formula for $\int x^m(lnx)^n dx, m\neq -1$

And n is an integer greater than 1. Hence evaluate,

$\int x^3(lnx)^2dx$

$$I=\int x^m(lnx)^n dx$$

$$I=(lnx)^n\left(\frac{x^{m+1}}{m+1}\right)-\int n(lnx)^{n-1}\frac{1}{x}.\left(\frac{x^{m+1}}{m+1}\right)dx$$

$$I=\frac{x^{m+1}(lnx)^n}{m+1}-\frac{n}{m+1}\int x^m(lnx)^{n-1}dx$$

Which is the required reduction formula.

Now put m=3 and n=2

$$I=\frac{x^{3+1}(lnx)^2}{3+1}-\frac{2}{3+1}\int x^3(lnx)^{2-1}dx$$

$$I=\frac{x^4(lnx)^2}{4}-\frac{2}{4}\int x^3lnxdx$$

Integrating by parts,

$$I=\frac{x^4(lnx)^2}{4}-\frac{2}{4}\left[(lnx)\left(\frac{x^4}{4}\right)-\int\left(\frac{1}{x}\right)\left(\frac{x^4}{4}\right)dx\right]$$

$$I=\frac{x^4(lnx)^2}{4}-\frac{1}{8}x^4lnx+\frac{1}{8}\int x^3\,dx$$

$$I=\frac{x^4(lnx)^2}{4}-\frac{1}{8}x^4lnx+\frac{1}{8}.\frac{x^4}{4}+c$$

Alternative method Q No. 9 $I=\int x^3\sqrt{x^2+1}\,dx$

$$I=\int x^2.x\sqrt{x^2+1}\,dx$$

Put $\sqrt{x^2+1}=z \quad\Rightarrow\quad x^2+1=z^2 \quad\Rightarrow\quad x^2=z^2-1$

$2xdx=2zdz \;\Rightarrow\; xdx=zdz$

So

$$I=\int(z^2-1)z.zdz$$

$$I=\int(z^4-z^2)dz$$

$$I=\frac{z^5}{5}-\frac{z^3}{3}+c$$

$$I=\frac{(x^2+1)^{\frac{5}{2}}}{5}-\frac{(x^2+1)^{\frac{3}{2}}}{3}+c$$

Alternative method Q No. 11 $I=\int e^x\frac{1-sinx}{1-cosx}dx$

$$I=\int e^x\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2sin^2\frac{x}{2}}dx$$

$$I=\int e^x\left[\frac{1}{2sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2sin^2\frac{x}{2}}\right]dx$$

$$I=\int e^x csc^2\frac{x}{2}.\frac{1}{2}dx-\int e^x\cot\frac{x}{2}dx$$

Applying by parts on first integral

$$I=(e^x)\left(-\cot\frac{x}{2}\right)-\int(e^x)\left(-\cot\frac{x}{2}\right)dx-\int e^x\cot\frac{x}{2}dx$$

$$I=-e^x\cot\frac{x}{2}+\int e^x\cot\frac{x}{2}dx\int e^x\cot\frac{x}{2}dx$$

$$I=-e^x\cot\frac{x}{2}+c$$

Techniques of Integration

EXERCISE 4.1

Q No. 1 $\int 0\,dx = c$

Q No. 2 $\int \sqrt{x}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}x^{\frac{3}{2}}$ + c

Q No. 3 $\int \frac{1+x}{x}dx = \int\left(\frac{1}{x}+\frac{x}{x}\right)dx = \int\frac{dx}{x} + \int dx$

$=\ln(x) + x + c$

Q No. 4 $\int \frac{x^2-1}{x^2+1}dx$

$= \int(1-\frac{2}{x^2+1})dx$ by long division

$= \int dx - 2\int\frac{dx}{x^2+1}$

$= x - 2\tan^{-1}x + c$

Q No. 5 $\int tan^2x\,dx = \int(sec^2x-1)\,dx$

$= \int sec^2x\,dx - \int dx$

$= tanx - x + c$

Q No. 6 $\int cot^2x\,dx = \int(cosec^2x-1)\,dx$

$= \int cosec^2x\,dx - \int dx$

$= -cotx - x + c$

Q No. 7 $\int cos^2x\,dx = \int\left(\frac{1+cos2x}{2}\right)dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int cos2xdx$

$= \frac{x}{2} + \frac{1}{2}\left(\frac{sin2x}{2}\right) + c$

Q No. 8 $\int sin^2x\,dx = \int\left(\frac{1-cos2x}{2}\right)dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int cos2xdx$

$= \frac{x}{2} - \frac{1}{2}\left(\frac{sin2x}{2}\right) + c$

Q No. 9 $\int\sqrt{1-cosx}\,dx = \int(\sqrt{2}sin\frac{x}{2})dx$

$= \sqrt{2}\int sin\frac{x}{2}dx$

$= \sqrt{2}\left(\frac{-cos\frac{x}{2}}{\frac{1}{2}}\right) + c$

$= -2\sqrt{2}cos\frac{x}{2} + c$

Q No. 10 $\int\sqrt{4-x^2}dx = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}$

Q No. 11 $\int\sqrt{4+x^2}dx = \frac{x}{2}\sqrt{4+x^2} + 2\sinh^{-1}\frac{x}{2}$

Q No. 12 $\int\sqrt{x^2-4}dx = \frac{x}{2}\sqrt{x^2-4} - 2\cosh^{-1}\frac{x}{2}$